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Problem Solving in Mathematics Education

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Content

Editor's Foreword

This volume presents the proceedings of the 23rd ProMath Conference, held at the Faculty of Science, University of Split, marking 25 years since the first ProMath meeting. Over the past quarter of a century, ProMath has provided a meeting place for a dedicated international group of researchers, educators, and students engaged in sustained dialogue on mathematics education.

The theme of the conference, Problem Solving in Mathematics Education, reflects a long-standing and shared research interest within the ProMath community. The contributions collected in this volume address problem solving from a range of theoretical and practical perspectives, offering insights that connect research with classroom practice.

The conference programme included a keynote lecture by Professor Tatjana Hodnik (Faculty of Education, University of Ljubljana, Slovenia), along with a carefully curated set of research presentations and posters addressing various aspects of problem solving in mathematics education. We were especially pleased by the participation of graduate students, whose contributions brought fresh perspectives to the discussions.

This proceedings volume includes peer-reviewed scientific papers based on talks presented at the conference. The papers were reviewed with respect to their content. Responsibility for the quality and correctness of the English language rests with the individual authors.

The organising committee would like to express its gratitude to all participants for their contributions and for coming to Split. We also acknowledge the members of the organising and programme committees, as well as the sponsors, whose support made the conference possible.

Finally, we would like to mention the contributions presented at the conference that are not represented by an article in this proceedings volume:

Research reports:

Ana Katalenić, *What is the problem? Multi-step contextual problems posed by prospective primary school teachers*

Hana Čadež, *Mathematical problems with no specific curricular school content as a source for developing inductive reasoning*

Poster:

Hajdi Pervan and Iva Šarić, *Jail break problem (for elementary school students)*

We hope that this volume will serve as a record of the meeting and encourage continued exchange and collaboration within this research community.

The Editors

MATHEMATICAL PROBLEMS IN DIFFERENT TERMINOLOGICAL FRAMEWORKS: THE COIN IN DIFFERENT CURRENCIES

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Abstract

The aim of the paper is to recall theoretical paradigms for mathematics education and to discuss the rhetoric of value polarisation in education. Brief background information will serve as a framework for the presentation of selected topics on mathematical problem solving, Bloom's Taxonomy (application 1956, application 2001) as one of the frameworks for positioning problem solving primarily in the role of applying mathematics under an umbrella of mathematical literacy and mathematical modelling. We will argue that the use of different terminological frameworks for mathematical problems in elementary school, such as mathematical literacy, influences the meaning of mathematics learning and problem solving. We must ask what kind of mathematics we want our students to learn, how educational policy shapes this content, and where the boundaries of mathematics truly lie. The aim of the paper is by no means to exhaust all the dimensions of the issue, but only some of them, which we have judged to have a significant impact on the development of the discipline of mathematics in the school context.

Keywords: *problem solving, mathematical literacy, Bloom taxonomy, theoretical paradigms, mathematics*

1. COMPLEMENTARITY AND INTERACTION OF THEORETICAL PARADIGMS THAT SHAPE (MATHEMATICS) EDUCATION

Throughout history, research on teaching and learning has been influenced by various theoretical learning paradigms based on different philosophical, mainly ontological and epistemological, premises. Due to differing assumptions about the existence of "reality", people, and their relationships, these paradigms explain the learning process in different ways, and consequently, different orientations for pedagogical work are derived from them.

The multi-paradigmatic approach to learning itself poses a unique challenge for researchers and teachers. However, an even greater problem arises when attempting to establish only one theoretical paradigm. In the Slovenian pedagogical context, such efforts are evident in debates where certain teaching methods, forms, and approaches are labelled as "modern", while those considered "forbidden" or undesirable are labelled as "traditional". Such debates operate within a discourse of value polarisation and generate an unscientific ideological discourse that significantly influences the understanding and implementation of teaching.

The transition from the 19th to the 20th century marks the beginning of the development of the didactics of mathematics. In response to the need for qualified mathematics teachers, the first mathematics didactics courses were established at universities worldwide at that time (Kilpatrick, 1992; Schubring, 1983). The first researchers to study the learning and teaching of mathematics were mathematicians and psychologists, which significantly influenced the design of research at the time and the understanding of what constitutes scientific work in mathematics education (Kilpatrick, 1992; Schubring, 1983).

Let us briefly review the theoretical paradigms that have shaped the teaching and learning of mathematics over time. Reviewing these theoretical paradigms will help us understand the changing role of mathematics in school contexts and in its teaching and learning.

Behaviourism

Originally, the theoretical paradigm of behaviourism (Kilpatrick, 2020; Schubring, 1983) had a significant influence and was the dominant paradigm in psychology from the early 20th century until the late 1950s, with proponents such as Pavlov, Watson and Skinner (De Corte et al., 1996; Schuh & Barab, 2008).

Individual emotions and thinking were deliberately excluded from studies (De Corte et al., 1996). To summarise this period, the following characteristics of behaviourism can be highlighted (Ertmer & Newby, 1993):

- focus on achieving visible and measurable results in behaviours, tasks and goals;
- identification of students' prior knowledge for lesson planning;
- linking content from the simple to the complex;
- reinforcing learning by rewarding performance;
- consolidation of knowledge and clear feedback.

Cognitivism

In the late 1950s, the previously dominant position of behaviourism was replaced by information processing theory (De Corte, 1996; Mayer, 1992, 1996; Schuh & Barab, 2008), which some authors equate with cognitivism (e.g. Schuh & Barab, 2008).

At the end of the 20th century, learners' beliefs, attitudes and values were also studied, as they were understood to have a significant impact on learning (Winne, 1985; Ertmer & Newby, 1993). To summarise, these features of cognitivism can be highlighted (Ertmer & Newby, 1993):

- emphasis on active participation in learning and the development of metacognition (self-planning, self-regulation);
- focus on hierarchy between concepts to illustrate underlying relationships;

- focus on structuring, organising and sequencing information for optimal processing;
- creating learning environments that enable and encourage students to make connections to prior knowledge (drawing on prior knowledge, using relevant examples and analogies).

Constructivism

In the 1970s and 1980s, a new understanding of learning emerged: learning as the construction of knowledge (Mayer, 1992, 1996). This led to differences in epistemological assumptions (Cobb, 2007; Ernest, 2018).

Constructivism became the dominant paradigm in mathematics education research, reaching its peak from the late 1980s to the mid-1990s, before interest declined worldwide.

Constructivism is essentially an epistemological theory (von Glasersfeld, 1984), a theory about how we learn, and is based on two central premises:

- the cognitive activity of the individual aims to establish regularities in the world of experience; and
- the identification of regularities presupposes experiences that the individual continuously evaluates in terms of equivalence and individual identity.

This conceptual starting point is extremely general and non-prescriptive (it is not a "guide to practice"). These two key points offer no clear conclusions for teacher education. Kirschner et al. (2006) argue that constructivist approaches often promote minimally guided instruction, which can leave students 'lost and frustrated' (p. 6). They warn that such methods overlook empirical evidence demonstrating the ineffectiveness of unguided learning environments. Trivialised versions of constructivism align closely with recent trends in instructional approaches such as group learning (emphasised in socio-constructivism), multiple pedagogies, child-centred learning, and learning styles, which contrast with traditional teaching, although the term "traditional" is never clearly defined.

In brief, constructivism can be described as follows (Ertmer & Newby, 1993):

- focus on learning in meaningful contexts;
- emphasis on the learner's control over their own learning and their ability to apply their knowledge;
- importance of representing mathematical ideas in different ways;
- problem solving;
- assessment focuses on transferable knowledge and skills.

Problem solving is emphasised by constructivism, aligning with Polya's view that problem solving should be the guiding principle of school mathematics teaching, rather than being limited to the activity of mathematicians. From this point onwards, problem solving has become a significant area of research in the teaching and learning.

Socio-cultural paradigms

At the beginning of the new millennium, interest in the role of social factors in the teaching and learning increased (Lerman, 2000). In contrast to epistemological, ontological, and knowledge debates that focused on individual knowledge acquisition and its relation to reality, the emphasis on social factors in mathematics education research has led to a so-called social turn. This concept refers to the emergence of theories that view thinking, reasoning, and meaning-making as consequences of social activities. Interest in researching the teaching and learning within a socio-cultural paradigm, which focuses on classroom climate and the socialisation process, has grown significantly (Gutiérrez, 2013).

2. THERE IS NO WAY OUT OF MULTIPLE PARADIGMS

What can we learn from different theoretical paradigms of learning and teaching mathematics? Firstly, they are complementary rather than mutually exclusive, each offering new insights into the processes of teaching and learning. Although many would like to elevate certain paradigms — often because they are fashionable or the latest trend — various agendas can create dichotomies in mathematics education (Hodnik & Krek, 2022). We do not deny contemporary knowledge, nor should we, and we cannot disregard previous findings that can be re-evaluated and given new meaning, but they should not be dismissed. Kirshner (2015) identifies as problematic the attempts to establish only one of the existing paradigms in education. This leads to conflict and confusion, as different learning paradigms underpin a range of instructional models that may seem contradictory and provide teachers with incompatible recommendations and guidelines for teaching (Kirshner, 2015). Proponents of a particular theoretical paradigm create slogans such as "active learning is important", "the learner must be the focus", "students should learn in groups", "the teacher is the director, the organiser of the lesson", and "the learner constructs the knowledge himself". These short formulas have the power to persuade, to encourage people to act for or against something, and to legitimise or discredit a particular pedagogical practice or theory (Kodelja, 2023). The more effective and persuasive these slogans are, the greater the danger that they will increasingly replace critical thinking (Kodelja, 2023).

Terms such as explanation, consolidation, abstract thinking, individual work, effort, productive struggle, and rigour are negatively evaluated and therefore removed from discourse and use. Incidentally, with Hattie's conceptualisation of the didactics of mathematics in Visible Learning for Mathematics, these dimensions of teaching are once again rediscovered and validated on the basis of extensive meta-research over time. Another shift in educational discourse is identified by Biesta (2022), who uses the term "learnification" – where the term teaching is replaced by learning in educational discourse. The shift from teaching to learning results from the influence of constructivist learning theories and sociocultural approaches to education, all of which argue in some way that everything depends on the activities of the "learner", with or without some "scaffolding". All of this has placed the "learner" at the centre of educational efforts and moved the teacher to the sidelines – as coach, facilitator, co-learner, friend, critical or otherwise, but rarely as teacher (Biesta, 2022). Another important point or mistake in the discussion of knowledge is the assumption that the pedagogical content of the learning experience is identical to the methods and processes (i.e. epistemology) of the discipline being studied, and the assumption that teaching should focus exclusively on application (which is also present in mathematics education – it is becoming increasingly important to emphasise or defend the value of learning mathematics in its application in different contexts of mathematical knowledge). "It is regrettable that current constructivist views have become ideological and often epistemologically opposed to the presentation and explanation of knowledge." (Kirschner et al., 2006, p. 18).

3. PROBLEM SOLVING

A mathematical problem is generally defined as a situation recognised by the solver as a challenge, for which he or she has no strategy for solving or cannot recall one. The solution to the problem has intellectual and/or practical value for the solver. As is well known, George Pólya in the 1950s was the first to identify problem solving as the focus of mathematics education.

In the years that followed, many scholars and researchers agreed that problem solving should be the main goal of mathematics education. Many researchers exploring various ideas related to problem solving (Cañadas & Castro, 2007; Manfreda Kolar et al., 2012; Mason et al., 2010; Pólya, 1945; Radford, 2008; Reid, 2002; Schoenfeld, 1985) advocate the promotion of problem solving in students. Nevertheless, problem solving is not accepted by teachers and developers of instructional materials in the same way as other mathematical topics, such as mastering written algorithms or solving equations. A brief analysis of mathematics curricula in most countries shows that students are expected to solve problems and reason mathematically. The ability of individuals to solve problems is related to the structure of their mental schema for problem solving (Hodnik Čadež & Manfreda Kolar, 2015), with the strength and compactness of this schema depending on the connectivity of components between schema groups (Eli et al., 2013). According to Islami et al. (2018), mathematical connections can be divided into two categories: 1) internal connections, i.e. connections between topics and mathematical elements, and 2) external connections, i.e. connections between mathematics and other subjects and between mathematics and daily life. This means that students demonstrate their mathematical understanding by making connections between mathematical concepts, facts, and procedures (Hiebert & Carpenter, 1992). Connections also help students remember skills and concepts and use them appropriately in problem solving (Eli et al., 2013). Students with good connections are better at solving mathematical problems, while those with poor connections are less successful (Baiduri et al., 2020). We have only mentioned some findings on problem solving, but there is a vast body of research on this topic. Curricula worldwide prescribe problem solving as the main classroom activity, but textbooks still lack sufficient problems, and there is no additional material containing appropriate problem situations that consider the mathematical content for students at a given level of knowledge. Hopefully, we have moved beyond the idea that students solve mathematical problems independently simply by using and appropriately applying their knowledge. In many assessments of students' mathematical knowledge (PISA, TIMSS, national exams in Slovenia), which are available online, it is evident that problem-solving skills are not among the top achievements. Heuristics is a general term for principles, methods, tools, and cognitive processes that enable problem solving (though they do not guarantee solutions). We distinguish between general heuristics (e.g. working backwards, analogy) and domain-specific heuristics (e.g. converting equations into equivalents, adding fractions). The investigation of heuristics for problem solving in mathematics was concentrated in the last 50 years of the 20th century.

We would like to take this opportunity to mention models for learning heuristics developed by Kilpatrick (1985), who identifies several pedagogical models for introducing heuristics into the mathematics classroom, each reflecting a different view of how problem-solving skills can be fostered in learners. The first model, osmosis, refers to the implicit and often unstructured transfer of problem-solving strategies through exposure to a rich learning environment. While this model may be effective for highly motivated learners, it risks leaving behind those who require more structured guidance.

In contrast, the memorisation model focuses on the explicit teaching of specific techniques tailored to particular types of problems. This approach often involves deliberate practice in recognising problems and applying appropriate heuristic tools, as well as formulating guiding questions that support problem solving.

The imitation model is based on the belief that students acquire problem-solving skills by closely observing and imitating the actions of more experienced individuals, such as teachers or peers. This learning supports the development of strategies through role models, but requires careful support to prevent mere memorisation.

A more collaborative approach is found in the cooperative model, where learners engage in problem-solving activities in small groups. This model utilises social interaction, encourages collective sense-making, and negotiation of different strategies.

Finally, the reflection model emphasises the importance of metacognition in problem solving. Here, heuristic knowledge is developed not only through active engagement with problems, but also through systematic reflection on one's own strategies and reasoning processes. This reflective practice encourages learners to evaluate the effectiveness of different approaches, promoting adaptability and deeper understanding. Taken together, these models represent a range of teaching strategies. Rather than viewing them as mutually exclusive, teachers should incorporate elements from each model when teaching problem solving, tailoring their approach to students' developmental levels, prior knowledge, and learning preferences. These heuristics remain relevant today, and we are convinced that, if properly implemented by the teacher, they can empower students to develop problem-solving strategies. We should not and cannot expect students to become good problem solvers entirely on their own. We do not deny that there may be other approaches to learning heuristics that are equally or even more relevant to classroom practice. Is it not quite obvious that all theoretical paradigms are interconnected in the above-mentioned models of learning heuristics?

A brief overview of problem solving in mathematics education suggests that problem solving is not a single activity and that there is extensive research on the topic. Teaching heuristics to students at different levels remains a challenge. Developing mental schemas for problem solving enables students to transfer strategies and heuristics to more complex problems. Most researchers in this field study the skills of individual problem solvers and focus less on the teaching of problem solving in the classroom, which is complex for both research and teaching. The question of teachers' knowledge and skills in problem solving, as well as the organisation of problem solving activities in the classroom, remains unresolved.

Nowadays, solving mathematical problems remains a focus, but in a slightly different way. With the increasing emphasis on developing generic transfer skills in education (as previously mentioned in the new educational discourse), there is a marked shift away from solving mathematical problems per se towards the development of 21st-century skills that primarily involve the application of knowledge, such as within the frameworks of mathematical literacy, mathematical modelling, STEM, and similar areas.

4. UNDERSTANDING THE CONCEPT OF "APPLICATION OF KNOWLEDGE" THROUGH TIME: PERSPECTIVE VIA BLOOM TAXONOMY, MATHEMATICAL LITERACY

Bloom's Taxonomy

To illustrate the evolving emphasis on applying knowledge in education, we turn to Bloom's Taxonomy — first published in 1956 and later revised in 2001 — as a clear example of how educational objectives have shifted over time.

Original Bloom's Taxonomy

The main purpose of Bloom's 1956 Taxonomy of Educational Objectives was to improve education by providing teachers with a common vocabulary to discuss curriculum and assessment issues more precisely, and to standardise teacher communication to enhance the sharing and development of curriculum and assessment methods.

The framework devised by Bloom and his collaborators consisted of six main categories (see Figure 1): knowledge, comprehension, application, analysis, synthesis, and evaluation. The categories following knowledge were presented as "skills and abilities", with knowledge as the necessary prerequisite for putting these skills and abilities into practice. A closer look at the application category in this taxonomy shows it consists of a definition — what application is (using information and materials to solve new problems or respond to specific situations), behavioural learning outcomes (applying learned material such as rules, methods, concepts, principles, laws, and theories), and teaching/learning methods (Bloom, 1956).

Revised Bloom Taxonomy

The current model of Bloom's Taxonomy, titled A Taxonomy for Teaching, Learning, and Assessment, focuses largely on activities that support learning and plays an important role in developing active curricula for teaching (Forehand, 2010). (We need to be careful when discussing "active..." – the meaning varies depending on the context in pedagogical texts.)

The revised taxonomy also introduced a more dynamic and learner-centred approach.

However, we must not forget (and this is often the case in interpretations of Bloom's taxonomy, especially in teacher training) that in the revised taxonomy, knowledge forms the basis of the main categories (which is not evident in the images usually presented as the revised Bloom's taxonomy), and the authors have created a separate taxonomy of knowledge: factual knowledge (knowledge of terminology, specific details and elements), conceptual knowledge (knowledge of classifications and categories, principles and generalisations, knowledge of theories, models and structures), procedural knowledge (knowledge of subject-specific skills and algorithms, subject-specific techniques and methods, and criteria for determining when to apply appropriate procedures), and metacognitive knowledge (strategic knowledge, knowledge of cognitive tasks, including appropriate contextual and conditional knowledge, and self-knowledge).

If we look more closely at the terms used in each taxonomy (see Figure 1), we see that nouns have been transformed into verbs, i.e. "application" has become "apply". How is the term "to apply" described in the revised Bloom's taxonomy?

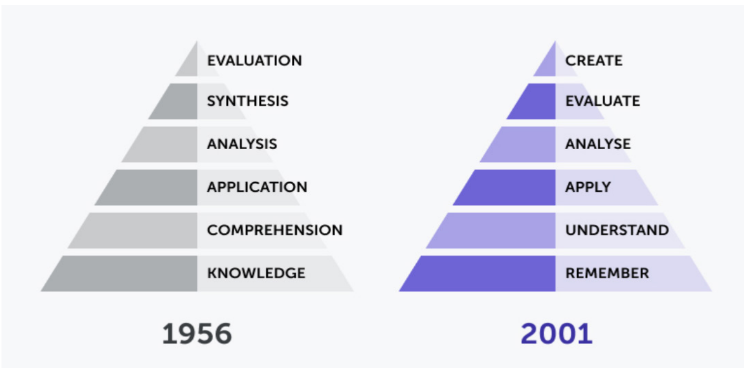

Figure 1: *Bloom taxonomy from 1956 and 2001*
(Taken from: https://www.valamis.com/hub/blooms-taxonomy)

"To apply" refers to using a concept in a new situation. Someone applies what was learned in the classroom to novel situations in the workplace. Examples given in the explanation of this term in Bloom's Taxonomy are mostly taken from professional contexts, such as using a manual to calculate an employee's holiday entitlement or applying laws of statistics to evaluate the reliability of a written test. Key words for "apply" include: applies, changes, computes, constructs, demonstrates, discovers, manipulates, modifies, operates, predicts, prepares, produces, relates, shows, solves, uses, calculates, illustrates, determines, models, performs, and presents. The prescribed technologies are: collaborative learning, creating a process, blog, and practice.

It is clear that the teacher's role is less present as in the explanation of "application" in the original Bloom's Taxonomy; in fact, there is no mention of what teachers do when pupils are applying knowledge in the revised taxonomy. Returning to the earlier discussion about value polarisation and the withdrawal of the teacher from the teaching role, this finding should come as no surprise.

Mathematical literacy

Alongside the application of knowledge, and almost simultaneously with the introduction of the revised Bloom's Taxonomy, a new dimension in mathematics education has emerged: mathematical literacy. Although the term was introduced much earlier, it gained prominence with the advent of PISA. The neologism "mathematical literacy" is one of several related terms used in English-language mathematics education research and policy discourse in connection with proposals to improve the teaching and learning of mathematics.

The 1989 NCTM standards articulate five general goals in pursuit of mathematical literacy for all students: 1) that they learn to value mathematics, 2) that they develop confidence in their mathematical abilities, 3) that they become mathematical problem solvers, 4) that they learn to communicate mathematically, and 5) that they learn to reason mathematically.

The first explicit attempt to define mathematical literacy can be found in the original OECD framework for PISA in 1999. This definition was modified slightly in subsequent PISA cycles (for the evolution of the mathematics framework over the years, see Stacey & Turner, 2015). The version of mathematical literacy for PISA 2015 reads (OECD 2017, p. 67) as follows:

"Mathematical literacy is an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make well-founded judgments and decisions needed by constructive, engaged and reflective citizens."

While "mathematical literacy", "quantitative literacy", and "numeracy" focus on mathematics as a tool for solving non-mathematical problems, "mathematical competence" (and "competencies") and "mathematical proficiency" concern what it means to be proficient in mathematics in general, including the ability to solve both mathematical and non-mathematical problems. The term "mathematical proficiency" (Kilpatrick, 2002) is intended to capture what successful mathematics learning means for all and is defined indirectly by five domains: conceptual understanding, procedural fluency, strategic competence, adaptive thinking, and productive disposition. Therefore, there are some reservations about the use and application of mathematical literacy in mathematics education:

- some reservations about the term "mathematical literacy" itself relate to the fact that there is no equivalent in several languages;
- another criticism is directed at attempts to define mathematical literacy in terms of transferable generic competencies or process skills, as such a concept tends to ignore the interests and values associated with posing and solving particular problems using mathematics.

Jablonka (2003) views mathematical literacy as a socially and culturally embedded practice and argues that notions of mathematical literacy vary according to the culture and values of those promoting it. De Lange (2003) also recognises the need to consider cultural differences when conceptualising mathematical literacy. There is no general agreement among mathematics educators about the contexts in which a mathematically literate citizen will or should engage, or for what purpose.

Based on a literature review, Jablonka (2003) identified five agendas on which conceptions of mathematical literacy are based: development of human capital (as exemplified by the concept used in the OECD PISA study), maintaining cultural identity, pursuing social change, creating environmental awareness, and evaluating mathematical applications. Each of these agendas reflects a different set of educational priorities and values, emphasising that mathematical literacy is a highly context-dependent and value-laden construct. They highlight the various intentions and interpretations associated with the term and suggest that a more nuanced and culturally sensitive understanding of mathematical competence may be necessary. However, is it solely about mathematics when we discuss the agendas outlined above (Jablonka, 2003)? Are we truly referring to mathematical literacy? Or are we, in fact, discussing the promotion of responsible citizenship? While we acknowledge the importance of fostering responsible citizenship, we find it problematic that the term “mathematical literacy” alters the meaning of learning mathematics and problem solving. Mathematics develops students’ notional thinking, which is essential for acquiring mathematical knowledge and for developing notional thinking more broadly. This is also important in other fields, and mathematics is present in most areas of human life and creativity.

5. CONCLUSION

Mathematics develops pupils’ conceptual thinking, enabling them to reason strictly using concepts, which is essential for acquiring mathematical knowledge. It also fosters general conceptual thinking, important for the development of rational thought and transferable to other sciences and areas of life. In this respect, it is similar to philosophical thinking in its conceptual rigour.

Accepting this view of the role of mathematics and teaching accordingly forms the foundation for preparing pupils to be responsible citizens, far more than using certain contextual problems under the umbrella of mathematical literacy, which may be irrelevant to the pupils or to reality. We would like to emphasise once again that objectives in the area of mathematical literacy can complement the mathematical knowledge base only to a very limited extent, but they cannot replace existing mathematical content.

It cannot be overlooked that various educational movements strongly influence our understanding of education, changing our perspective on knowledge (content, skills and values), and introducing different ‘innovations’. In this context, mathematics is one of the most affected subjects.

It is therefore necessary to think critically about changes, theories and concepts we use, and to insist on developing the knowledge that distinguishes the discipline of mathematics — there should be no doubt about what that is for mathematics.

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EXPLORING THE GAP BETWEEN THEORY AND PRACTICE IN MATHEMATICS PROBLEM-SOLVING TEACHING - PRACTICING MATHEMATICS TEACHERS AND THE DISCOVERY LEARNING

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Abstract

George Pólya's ideas relating to mathematical problem-solving teaching and learning have influenced Hungarian mathematics teaching. One of these ideas is discovery learning. In our study, we asked some mathematics teachers about their opinions and experiences related to discovery learning. In this paper, we analyze the theoretical background of discovery learning (minimally guided learning) and then compare them with real teaching experiences. Teachers' practical experiences suggest that this method is effective mainly under specific conditions.

Keywords: *fully guided vs minimally guided instruction, working memory, long-term memory, novice vs expert students*

1. INTRODUCTION

One of the elements of mathematics teaching in the 21st century - emphasized by many - is the active learning process, "active learning" in English or "Arbeitsprinzip" in German. *"Active learning engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert."* (Freeman et al., 2014, p. 8413-8414). It is important to activate the students in the lessons so that they do not just sit passively through the 45 minutes. In order to acquire mathematical concepts in the long term, it is essential that students actively practice with the given concepts. *"Socrates already said that we learn best what we struggle and work with ourselves, the secrets of which we discover ourselves."* (Pólya, 1968, p. 115). In mathematics education discovery learning is an inquiry-based, constructivist instructional approach in which learners actively explore materials or problems, generate hypotheses or procedures, and construct and test their own understanding rather than being presented the final formalisation by the teacher.

If we talk about the well-known and ongoing theories of active teaching and learning, we should also talk about what the real teaching practice shows in connection with them. On CERME 13 Budapest there was a panel discussion about the gap between theory and practice in mathematics education. The consensus was that this gap is very large. De Ponte (2023) emphasized that a three-pole system should be considered: research – practice – school policy. Summarizing our experiences in teacher training: the very hard task is to reach the practicing teachers to collaborate with them from researchers' side. It is not easy to establish cooperation between researchers and practicing teachers for the purpose of connecting theory and practice (Gutiérrez, 2017, Opstoel et al., 2024, Wijarwadi et al., 2025).

In our paper, we analyze the role of discovery learning in mathematics teaching practice, because a lot of didactical researchers from the 20th century (Bruner, 1961; Pólya, 1968) and some new researchers also (Brunstein et al., 2009; Jatisunda et al., 2020; Diano et al., 2021; Stacey,

2022) claimed that discovery learning has a great effect on students' performance in mathematics, make students motivated towards learning, that is why this form should be dominated in teachers' mathematics lessons. However, in practice, many teachers still prefer to conduct lessons that are kept under an appropriate level of guidance.

In the following we introduce the main concepts which are important to active learning. First, the most important one is discovery learning. We shortly summarize the main characteristics, then we show some research results arguing with the effectiveness or uneffectivness of this method of learning. Then, we sum up the most recent point of view about how our long-term memory works. At last, we list some – research-based – disadvantages of the minimally guided learning process.

2. THEORETICAL BACKGROUND

2.1. Discovery learning - origins

One of the well-known origins of the concept of discovery learning is rooted back to Bruner (1961). Bruner argues that students learn more effectively and retain information longer when they discover it themselves rather than being directly taught.

If we try to define it, according to researchers' views in this topic, we can state that discovery learning is a constructivist, inquiry-oriented approach in which students actively build knowledge by exploring a subject, experimenting, and drawing conclusions instead of receiving rules directly from the teacher. As Honomichl and Chen (2012) note, it emphasizes engaged participation and reflective investigation, typically in problem-solving settings, where learners use their prior experiences to develop strategies and grasp underlying concepts.

2.1.1. Hungarian representatives of discovery learning

The discovery learning trend has a long tradition in Hungarian mathematics education as well. One of its well-known representatives is George Pólya. As Pólya writes: *“Learning begins with activity and perception, from this it passes into words and concepts and finally leads to the right way of thinking.”* (Pólya, 1968, p. 114). According to his thoughts, learning as a process cannot be only passive and repetitive, but must necessarily be an active process. In this process, the teacher has an important role in facilitating his students in the direction of active learning. Active learning increases students’ cognitive activity, which helps long-term memory to store new concepts and knowledge elements. In order to achieve this, discovery mathematics teaching can help students the most (Pólya, 1968).

Another respected mathematics teacher is Lajos Pósa, who developed his own teaching style over the past decades, the Pósa method (Urbanski et al., 2022), based on the students’ creativity and interest in problem-solving. Pósa uses carefully designed problem threads – sequences of related tasks that build on each other like building blocks. Students explore problems collaboratively, discover key ideas and develop reasoning and problem-solving skills. With this method, the teacher encourages his students to dig into the depths of a particular problem, i.e. to get to know and work around the given problem thoroughly. The teacher provides enough time for his students to find an answer, a correct definition, or interesting additional questions related to the problem. According to these characteristics, we can state that, the Pósa method is another Hungarian representative of discovery learning (Urbanski et al., 2022).

The Pósa method also basically corresponds to the Pólya principles, i.e. it aims to promote students’ independent thinking and increase their cognitive activity. Pósa’s method is particularly effective among gifted students (expert problem-solvers), at least in terms of elaborated and interconnected problem areas. A critical point of the method is the time factor. As mentioned earlier, this method provides enough time for students to delve deeper into solving a problem. However, *“less gifted children find it more difficult to concentrate for extended periods, and they are quicker to regard it as a failure on their part if they are not able to solve a problem”* (Győri & Juhász, 2017, p. 96).

2.1.2. Different types of discovery learning

According to Mayer (2004) discovery learning is not a single method but a spectrum ranging from *pure discovery* (unguided discovery: learning without teacher intervention) to *guided discovery* (scaffolding tailored to the learner's needs). Our study mainly focuses on minimally guided discovery approach, which means students explore mostly on their own, with very limited help from the teacher. In terms of Mayer's spectrum, minimally guided discovery is almost pure discovery. Mayer‘s study (2004) showed that – with several examples of earlier research – pure discovery has more disadvantages than benefits, because in unguided explorations most of the students – who does not have enough experience or talent – can easily "get lost".

2.1.3. Constructivist fallacy

The minimally guided instruction is based on constructivist learning theory. Constructivism, a widely accepted theory of learning, suggests that learners actively build their understanding of the world by engaging in cognitive processes. However, the active cognitive engagement needed for knowledge construction can occur through various activities, such as reading books, attending lectures, or observing a teacher demonstrate an experiment while explaining it. Constructivist teaching wants students to be critically thinking, motivated and independent learners (Daodu et al., 2024). Learning involves students actively building knowledge, but proponents of minimally guided instruction often confuse constructivism, which describes how we learn, with a method of teaching. Simply withholding information from students does not support the effective construction of knowledge (Clark et al., 2012).

In short, constructivism explains how people learn – that learners actively build knowledge based on experience – but it does not mean that simply giving students minimal guidance or letting them "discover" everything on their own will automatically lead to effective learning.

Overall, discovery learning has a long tradition in both Hungarian and other nations' mathematics education system with several advantages (in certain groups with certain types of students), these last paragraphs show us that it also has disadvantages too. We describe some of its main disadvantages in 2.4. focusing on minimally guided discovery learning.

2.2. Realizing Pólya's ideas

Schoenfeld (2022) writes about his own teaching experience in connection with Pólya's ideas. In the beginning – just as other mathematics teachers – Schoenfeld was impressed by those ideas mentioned in Pólya's books, like "How to solve it" and others. Then time passed. Unfortunately, nowadays teaching practice showed that those ideas of Pólya "*weren't specific enough to help teachers make the ideas come alive in the classroom.*" (Schoenfeld, 2022, p. 159). Basically, he stated that, in his opinion, teachers must take greater control over their lessons, because for most students, minimally guided teaching methods are just not effective. They often benefit from more structured and explicit forms of instruction. As he writes, *"Patterns are there to be found. But, one has to have one's eyes open."* (Schoenfeld, 2022, p. 158).

In the following we emphasize and analyze the human memory structure, the role of working and long-term memory, the cognitive load and its consequences on mathematical problem-solving teaching, to demonstrate why, in our opinion, minimally guided instruction hardly can work at "less gifted" students (later called: novices).

2.3. The human memory structure

Most neuroscientists accept Baddeley's model (1974) of memory structures: sensory memory, working memory, and long-term memory. The relationship between these parts of our memory is shown in the *Figure 1*. (Digiesi et al., 2020) Sensory memory refers to the very brief storage of raw information coming from the sensory organs (e.g., vision, hearing, touch). This type of memory operates automatically, meaning it does not require conscious attention, and it can retain information for only a few hundred milliseconds up to one or two seconds. Baddeley and Hitch (1974) emphasize that sensory memory is not part of working memory; rather, it serves as the input channel for it. The two additional parts of the memory model that are relevant to this paper are described in detail below.

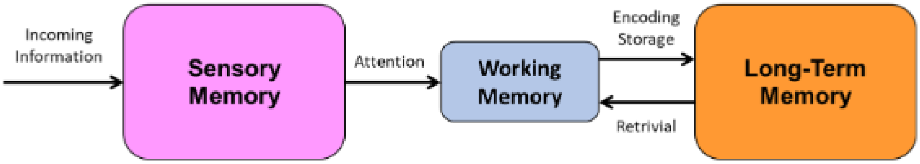

Figure 1. *Memory structure (Digiesi et al., 2020, p. 3)*

2.3.1. Working Memory (WM)

WM is called the workbench of our brain; it is the active problem space. Our WM constructs plans, uses transformation strategies, analogies, and metaphors, brings together things in thought, abstracts and externalizes mental representations. WM has a very limited capacity holding 7 ± 2 information units. An information unit refers to a single chunk of information that working memory can handle as one item, regardless of how complex it actually is. Its time limit is 18 - 30 sec without rehearsal. There is a processing limit, too: if we organize, contrast, compare, and work on information, only two or three items of information can be processed in parallel. There may be huge differences relating to WM capacity between students.

2.3.2. Long-Term Memory (LTM)

Long-term memory (LTM) stores information in the form of schemas, which are abstract, structured, and dynamic mental representations of knowledge. When a schema becomes automatic, it represents a skill or procedure that can be used without placing additional demands on working memory (WM). This has an important implication: by retrieving relevant schemas from LTM, we can effectively extend the capacity of WM, since each schema can function as a single unit of information despite containing multiple elements. Within WM, new information is processed in relation to existing schemas: it may be incorporated into them, similar schemas may be generated and modified, or entirely new schemas may be created and encoded back into LTM. A key difference between experts and novices is that experts possess a large number of well-developed schemas, which they can efficiently apply to problem-solving tasks, whereas novices have fewer, more isolated schemas. LTM has no known strict limits in terms of capacity or duration.

The role of LTM in human cognition has gone through a huge alteration following the research of the past decades. Today, it is not considered a passive repository of isolated fragments of information, nor does it fill a part of human cognitive architecture with only a peripheral effect. *"Rather, long-term memory is now viewed as the central, dominant structure of human cognition. Everything we see, hear, and think about is critically dependent on and influenced by our long-term memory."* (Kirschner et al., 2006, p. 76).

2.4. Problems with minimally guided problem-solving teaching in the classroom context

2.4.1. Difference between expert and novice learners

Expert problem solvers in mathematics rely on their extensive experience, which is stored in LTM as organized patterns of knowledge and methods, often referred to as mental schemas. These schemas allow them to quickly access and apply previously learned procedures and strategies when faced with new problems. Essentially, our LTM serves as a vast repository of knowledge that underpins our cognitive abilities. The primary goal of education is to enrich this knowledge base by adding new information and techniques to our LTM. If learning doesn't result in the accumulation of new information in memory, then true learning has not occurred (Clark et al., 2012, p. 9).

Novice learners typically lack the relevant concepts or methods stored in their LTM. According to cognitive load theory researchers, when teaching new material to novices, direct and clear guidance is more beneficial than offering incomplete guidance (Clark et al., 2012, p. 8). Teachers are more successful when they combine explicit instructions with opportunities for practice and feedback. Independent problem-solving tasks are most useful not for encouraging discoveries, but for reinforcing recently learned content and skills.

There are several empirical studies that showed: *"… for everyone but experts, partial guidance during instruction is significantly less effective than full guidance."* (Clark et al. 2012, p. 7). One of the main reasons for this is that, unlike experts, novices do not possess sufficient or well-structured prior knowledge in their long-term memory (LTM). As a result, due to the limited capacity of working memory (WM), novices often experience excessive cognitive load (Kirschner et al., 2006).

2.4.2. Research of minimally guided instruction

Richard Mayer (2004) has collected several pieces of evidence from decades of studies to support direct teaching methods over minimally guided teaching. Every decade from the 1950s to the late 1980s and present days, research provided evidence that minimally guided, problem-based discovery learning does not work in the useful way it should, mainly because of the above-mentioned reasons.

In his article (Mayer, 2004) he emphasized that he has found "three strikes against pure discovery". In short, these three strikes claimed that well guided discovery learning methods have led children to better understanding of concepts and they can transfer the learnt procedures better to other problems. The teacher's support and feedback are also an important factor in students' learning procedure, because in minimally or pure discovery children may not be aware of their mistakes or the critical features of tasks, and in these cases, misconceptions can easily be developed in their minds. Pure discovery method might support students' simple skills, but study showed that children struggled with deeper, more abstract concepts unless provided with structured guidance (Mayer, 2004).

According to Clark et al. (2012), many problems can arise during ordinary lessons when the teacher takes a back seat and applies minimally guided instructions. The four most important points described are presented below.

Firstly, since the most prepared, smartest learners are the most successful in discovering new knowledge, they are the ones who answer first to the teacher's questions, always just ahead of their classmates. This situation makes many other students frustrated, and undermotivated in discovering something new.

Secondly, frustrated and undermotivated students miss the train of thought of smart students who make quick discoveries, so slower students don't discover what they should. This kind of classroom situation does not provide the optimal learning environment. We know that emotions are a crucial factor in the process of learning. It is stated that expert problem-solvers usually control better their emotions than novices (Schoenfeld, 1985). That is why the teacher has a chance to solve this problem by choosing the most useful teaching method for the class.

Thirdly, since the teacher's role is completely overshadowed, many students may make a wrong discovery, so the formed misconception will hinder the acquisition of later concepts because of the spiral structure of mathematics teaching and learning. Students must have strong and well-developed schemes of their mathematical concepts so that the teacher can build the new concepts based on the previous ones, moving from lower to higher structural level of understanding (Sfard, 1991).

Fourthly, even if students work on problems that can be successfully completed by all students, explicit guidance is a more effective teaching method from the point of view of all students than minimally guided instruction. By efficiency, Clark primarily means the time factor. A topic that could be taught in 40 minutes through direct instruction may take several hours through minimally guided discovery. Based on teachers' experience in mathematics teaching, the time factor might be the most critical factor of all (Clark et al., 2012).

3. METHOD

Our research is a case study; we asked 7 practicing mathematics teachers about minimally guided discovery teaching and learning process. These 7 mathematics teachers teach in primary or secondary schools. We asked teachers working in our environment to be part of our sample. Teachers were selected based on their willingness to participate and their varying years of experience to ensure a diverse sample. Among them, the five high school teachers have 3, 5, 20, 30, and 40 years of teaching experience, while the two elementary school teachers have 30 and 35 years of experience. All of them have taught and continue to teach students with average performance in mathematics.

The data were collected through open-ended interviews. Teachers were first asked the following question. What do you think about teaching minimally guided discovery learning in maths? (The concept mentioned in the question was clarified, and we used the definition provided in the theoretical overview of the present article.) They were asked to give their views in writing based on their teaching experiences. The next step was the teachers receiving scientific material about the theoretical background of fully guided versus minimally guided mathematics instruction. Finally, they commented on this material based on their own teaching experiences.

The teachers' responses were analysed using qualitative content analysis to identify the factors that they mentioned as advantages or disadvantages of discovery learning. Both inductive coding, emerging from the data, and deductive elements, informed by the theoretical framework of guided versus minimally guided instruction, were applied. The analysis aimed to uncover recurring themes and patterns in teachers' reasoning, thereby providing insight into their perspectives on minimally guided discovery learning and guided instruction in mathematics.

4. RESULTS AND DISCUSSION PRACTICING TEACHERS ABOUT DISCOVERY LEARNING AND TEACHERS' REFLECTIONS

The teachers' opinions mention four main factors that they believe strongly influence the applicability of minimally guided discovery learning in mathematics lessons. The three authors independently analyzed the teachers' opinions and reflections, then developed a common coding scheme and re-analyzed the teachers' responses. This process resulted in the categories derived from both inductive and deductive coding:

1) the size of the group,
2) the time available,
3) students' level of mathematical development (expert vs novice),
4) the methodological support.

In the following four subsections, we will discuss these categories, illustrating them with a few quotes from teachers.

4.1. The size of the group

"How can I share my attention more among the students?"

"In a demonstration class, the teacher can work with twelve children."

These statements illustrate teachers' perception that discovery learning requires intensive monitoring and individualized feedback, which becomes impractical in large classes of 20–30 students. The problem is not only quantitative but also qualitative: the teacher's role shifts from explaining to facilitating (Pólya, 1968), which is cognitively demanding when many students progress at different speeds. This finding supports Clark et al. (2012), who emphasized that limited guidance can create unequal participation and learning outcomes within heterogeneous groups.

4.2. The time available

"In addition to the number of lessons and the amount of material, discovery learning can only fit into the curriculum for one or two tasks."

"Almost every lesson contains new material, with a quick summary at the end of the lesson."

"It's just that in 3–5 lessons a week, you don't have time to discover everything you need to be successful."

”Moreover, the method requires a lot of time not only in class but also in preparation.”

These remarks reflect a structural issue: teachers are expected to cover extensive curricula while maintaining an exam-oriented pace. Discovery activities, which require exploration, reflection, and discussion, conflict with this schedule. Teachers thus see the approach as ideal in principle but unrealistic in practice. This theme echoes Mayer’s (2004) and Clark et al. (2012) argument that the time efficiency of direct instruction remains a major advantage in formal education contexts.

4.3. Students’ level of mathematical development (expert vs novice)

“In the weaker groups, my attempts are less successful.”

“I like to use it in the best groups; the children like it, but in the weaker groups, we use a different approach.”

Teachers repeatedly distinguished between “stronger” and “weaker” groups, associating discovery learning with expert students. As Kirschner et al. (2006) mention, expert learners can rely on existing mental schemas stored in long-term memory, whereas novices experience cognitive overload when left to discover new concepts independently. The comments indicate that teachers intuitively recognize this cognitive principle, even without formal theoretical framing.

In the case of novice students, algorithmic and procedural activities (Sfard, 1991) should be used first, as they not only help to build mathematical self-confidence through experience, but also contribute to the development of conceptual understanding. Also, novices require more (positive) feedback from the teacher (Mayer, 2004). After gaining sufficient practice and confidence, novice learners will be more capable of effectively discovering mathematical concepts.

4.4. The methodological support

"I feel that what we need most is practical advice."

"There are no tried and tested, detailed lesson plans or materials for teachers on the method."

Here, teachers express not resistance but conditional openness. They would like to use discovery learning more often but feel unsupported by the educational system. The lack of structured materials and methodological guidance discourages experimentation. This finding aligns with Da Ponte (2023), who described how the absence of shared tools and institutional backing widens the gap between research and classroom implementation.

4.5. Teachers' reflections on the theoretical materials

This section summarizes the reflections on the theoretical material sent to teachers.

"It is difficult to find the invisible boundary that separates students who can be called 'experts' from novices."

"Fully guided instruction is necessary for novices. I think I've figured that out too."

"Theory is far from practice. At the end of many training sessions, I feel that yes, this is all very nice, but with 20–25 mostly unmotivated children, it is hardly possible."

These reflective excerpts show a critical awareness among teachers. They do not reject theory but reinterpret it through their own classroom realities. Most participants acknowledge that different levels of guidance are needed for different learners, supporting a continuum model rather than a dichotomy between discovery and direct instruction. Teachers' comments also highlight emotional dimensions – frustration, fatigue, and pressure – which mirror Schoenfeld's (2022) observation that teachers must actively help students "open their eyes" rather than expect discovery to occur spontaneously.

4.6. Summary of thematic patterns

Across all responses, teachers appreciate the motivational value of discovery learning but identify structural and cognitive constraints that limit its classroom feasibility. The method appears to work best in small, advanced groups with ample time. These findings reinforce the need for a balanced, hybrid approach – combining guided instruction with discovery elements – rather than a purely minimally guided model.

5. CONCLUSION

This study aimed to explore how practicing mathematics teachers perceive and apply discovery learning, and to what extent theoretical ideas align with classroom realities. While our sample size was limited, the findings highlight several consistent patterns that deserve attention.

Teachers were generally open to discovery learning because of its motivational and cognitive potential. However, their reflections and experiences revealed that, under current school conditions, the method is often difficult to apply effectively. Four main factors emerged as barriers: large class sizes, limited instructional time, students' varying levels of mathematical development, and a lack of methodological support.

Our results are consistent with previous research (e.g., Clark et al., 2012; Mayer, 2004), which also points out that minimally guided discovery learning may not be suitable for most learners, especially for novices. For these students, insufficient guidance can lead to frustration and lower motivation. In such cases, structured, fully guided instruction – including worked examples and scaffolded practice – can support confidence and conceptual understanding. Once students gain sufficient experience, more problem-based discovery tasks can be introduced effectively.

Teachers also emphasized that lesson preparation for discovery-based activities is time-consuming, and existing resources are limited. Providing ready-to-use, research-based teaching materials, would help teachers integrate discovery learning in a more realistic and sustainable way.

Therefore, rather than abandoning discovery learning altogether, it should be reconsidered and adapted to several kind of group of students. Hybrid models that combine guided instruction with exploratory elements may provide a more balanced approach. As Schoenfeld (2022) reminds us, *"Patterns are there to be found, but one has to have one's eyes open."* (Schoenfeld, 2022, p. 158). Helping students open their eyes through appropriate support, rather than simply explaining patterns to them, might be the real path forward.

In conclusion, bridging the gap between theory and practice requires coordinated efforts from researchers, teachers, and policymakers. Future studies should explore how technological tools and collaborative professional development can make discovery-based methods more accessible and effective in everyday mathematics teaching in different kind of group of students.

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GEOMETRY EDUCATION IN MOTION: ENHANCING TEACHER ATTITUDES FOR STUDENT PROBLEM-SOLVING SKILLS THROUGH COLLABORATIVE ACTION RESEARCH

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Abstract

This study investigates the impact of Collaborative Action Research on enhancing teacher attitudes and instructional practices in geometry education through Development of Geometrical Thinking program, implemented with four Croatian mathematics teachers and 88 7th-grade students, prioritized cognitive apprehension and student-centred strategies to foster problem-solving skills. Quantitative analyses revealed significant improvements in students' geometric thinking, while qualitative findings highlighted shifts in teacher attitudes, particularly toward adopting hands-on, inquiry-based approaches. By bridging research and practice, this study underscores the transformative potential of Collaborative Action Research in advancing geometry instruction and fostering professional growth among educators.

Keywords: *geometric figure apprehension, mathematics education, professional development, teacher attitudes*

1. INTRODUCTION

Teachers are instrumental in shaping both students' mathematical knowledge and their attitudes toward learning. In geometry, where visualization, spatial reasoning, and abstract thinking are essential, effective teaching requires not only subject expertise but also reflective practice.

Reflective practice — through which teachers critically examine their instructional decisions, classroom interactions, and student learning — enables them to adapt strategies to meet diverse student needs, fostering deeper conceptual understanding and stronger problem-solving skills (Schoenfeld, 2019).

Attitudes are particularly important in this process. Positive teacher attitudes encourage student-centred, evidence-based approaches that enhance engagement and learning outcomes, while negative attitudes can limit innovation and hinder conceptual growth (Guskey, 2002; Reid & Ali, 2020). Since attitudes are shaped by beliefs, emotions, and behaviours, they are not static but can evolve through professional reflection and collaboration.

Collaborative Action Research (CAR) provides a practical framework for reshaping teacher attitudes and instructional practices. By engaging teachers in iterative cycles of planning, action, observation, and reflection, CAR empowers them to integrate theoretical insights with classroom practice, fostering meaningful changes in both teaching and learning (McNiff & Whitehead, 2011). This study investigates how CAR, implemented through the Development of Geometrical Thinking (DGT) program, supports teachers in addressing challenges in geometry instruction and illustrates how changes in teacher attitudes and practices are reflected in student outcomes.

2. THEORETICAL BACKGROUND

Attitudes play a pivotal role in shaping teaching and learning, especially in subjects like mathematics, where cognitive and emotional barriers can significantly impact outcomes. Eagly and Chaiken (1993) define attitudes as psychological tendencies expressed through evaluations of objects with varying degrees of favour or disfavour. In educational contexts, teacher attitudes influence how subject matter is approached, how teaching is delivered, and how learners engage with the content. These attitudes are particularly crucial in shaping the learning environment, guiding instructional decisions, and ultimately determining student success.

Attitudes are complex and multidimensional, comprising three interrelated components: cognitive, affective, and behavioural. From a teacher's perspective, the cognitive dimension refers to beliefs, such as their perception of the relevance or difficulty of geometry. The affective dimension encompasses emotional responses, such as enthusiasm or anxiety, which shape how teachers feel about their subject, or some topics and teaching. The behavioural dimension is reflected in observable actions, including the extent to which teachers are willing to adopt innovative strategies or avoid challenging topics. These components interact dynamically, influencing how teachers engage with new teaching methods and professional development opportunities (Hannula, 2002; Reid, 2015; Aquilina et al., 2024). Moreover, attitudes toward teaching and learning have been found to mediate emotions and beliefs, serving as a bridge between these domains (Di Martino & Zan, 2011).

Measuring attitudes presents a unique challenge due to their subjective and internal nature. Indirect tools are often employed to capture the cognitive, affective, and behavioural dimensions. Reid's (2015, p. 265) holistic approach to measuring attitudes recognizes the complexity and multidimensionality of attitudes, proposing a framework that incorporates oral, written, and observational methods to capture these dimensions. This approach is particularly valuable, as without a clear understanding of students' and teachers' attitudes, it is challenging to implement effective educational strategies or interventions. By focusing on attitudes, educators and researchers can identify areas for improvement in teaching and learning, fostering a more supportive and effective educational environment. Oral methods, such as interviews and focus groups, allow for an in-depth exploration of the underlying beliefs and emotions that shape attitudes. Written approaches, including survey questionnaires with Likert scales and semantic differentials, provide structured means to assess agreement with specific statements or rate perceptions using opposing adjectives. These methods allow for a more direct evaluation of attitudes through written responses. Observational methods, or "doing," offer valuable insights into how attitudes manifest in practice by observing behaviours in real-world settings. The integration of these methods ensures a comprehensive understanding of attitudes and their influence on instructional practices (Reid, 2015). In particular, Hannula (2002) argues that emotions, values, and expectations are core dimensions influencing teachers' and students' attitudes toward mathematics. The teacher's enjoyment of mathematics has also been shown to correlate with positive classroom practices and attitudes toward student struggles (Russo et al., 2020).

Although attitudes can be resistant to change, they are not immutable. Targeted interventions, reflective practices, and meaningful professional experiences can influence and reshape teacher attitudes. Cognitive reframing encourages teachers to critically examine and adjust their beliefs about teaching and learning, while affective engagement fosters positive emotions that reduce anxiety and inspire enthusiasm (Vidić & Đuranović, 2020). Behavioural reinforcement, achieved through the observable success of new teaching strategies, further motivates teachers to adopt and sustain innovative practices. This is where frameworks like CAR come into play, providing a structured and supportive approach for teachers to engage in the process of attitude change and improvement in instructional practices.

The interplay between research and practice, as Schoenfeld (2019) advocates, underscores the importance of frameworks like CAR, which emphasizes collaboration between researchers and teacher practitioners to address educational challenges. CAR engages teachers in iterative cycles of planning, action, observation, and reflection, fostering productive synergy between evidence-based insights and practical applications (Bonner, 2006). By enabling teachers to test scientific findings and adapt them through refined strategies in real classroom settings, CAR bridges the gap between research and practice, promoting effective and sustainable changes in attitudes and teaching methods. The interplay between research and practice, as Schoenfeld (2019) advocates, underscores the importance of frameworks like CAR, which foster collaboration between researchers and teachers to address educational challenges. CAR engages teachers in iterative cycles of planning, action, observation, and reflection, creating a productive synergy between evidence-based insights and classroom practice (Bonner, 2006; Yuan & Lee, 2015). By testing scientific findings and refining strategies in real settings, teachers can bridge the gap between research and practice, promoting meaningful and sustainable changes in teaching methods and professional attitudes.

In geometry education, CAR enables teachers to adopt more student-centred approaches and enhance their geometric thinking. This aligns with research emphasizing collaboration in teacher-led action research (Bruce et al., 2011) and with theoretical frameworks for geometric thinking — Duval's Geometric Figure Apprehension (GFA; Duval, 1995) and van Hiele's levels of geometric thinking (VH; van Hiele, 1986) — which highlight hands-on activities, visualization, reasoning, and problem-solving. Through CAR, teachers can apply these frameworks to transform their practices and create engaging, theory-informed learning environments (Wright, 2021).

Building on these insights, the present study examines how CAR can serve as a transformative tool for reshaping teacher attitudes and enhancing instructional practices in geometry education. Although much of the existing literature emphasizes students' attitudes, research on teachers' attitudes — particularly in geometry — remains scarce, especially in Croatia where CAR is largely underutilized (Bognar, 2011).

3. RESEARCH QUESTIONS

The research seeks to address the following key questions:

1. How does the CAR process influence teachers' attitudes toward geometry and their instructional practices?
2. In what ways do changes in teachers' attitudes and teaching methods, as a result of CAR, impact students' understanding of geometric concepts and their problem-solving skills?

4. METHODOLOGY

4.1. Research design

This study is part of broader research (Antunović, 2024a) that employed a quasi-experimental mixed-methods design to examine the impact of the DGT program. An experimental group of teachers participated in the program, while a control group teachers continued with usual instructional practices, as illustrated in Figure 1, which depicts the overall research design and group assignments. Although elements of CAR were incorporated as a professional development tool (McNiff & Whitehead, 2011), the study remained primarily intervention-based, ensuring both rigor and practical relevance (Ivankova & Wingo, 2018; Martí, 2016). The present paper focuses specifically on teachers' attitudinal shifts toward geometry, with summarized qualitative findings complementing the quantitative analysis.

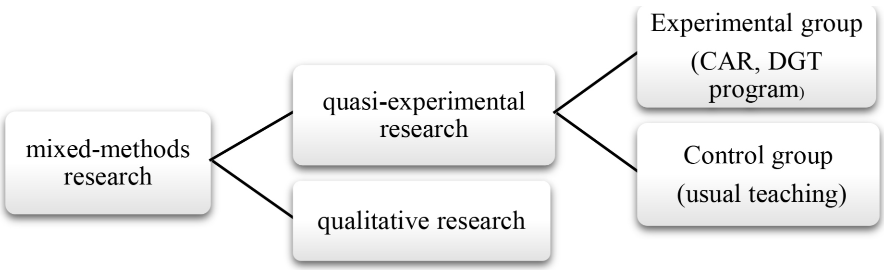


Figure 1. *Graphical representation of the specificities of the research approach (Antunović, 2024b)*

4.2. Participants

The participants in this study were four in-service mathematics teachers and four seventh-grade classes from ordinary urban primary schools in Croatia, without specialized mathematics programs or selective admission criteria, totalling 88 students. The experimental group (EG) comprised two classes (EG1 and EG2) and their respective teachers (ET, i.e. ET1 and ET2), who participated in the DGT program. The remaining two classes (CG1 and CG2) formed the control group (CG), with their teachers (CT, i.e. CT1 and CT2) continuing with usual instructional methods. The four teachers were selected for their willingness to collaborate and comparable classroom resources, with teaching experience ranging from 15 years (CT2) to over 20 years (ET1, ET2, and CT1). Seventh-grade students were chosen because previous research on van Hiele levels indicates that most students at this age (12–13 years) are at the first two levels of geometric thinking (e.g., Škrbec & Hodnik Čadež, 2015). At this stage of cognitive development, students also enter the operational phase, enabling them to understand and explain abstract mathematical concepts and ideas, meaning they can reach the third van Hiele level, the level of informal deduction.

4.3. Data Collection and Analysis

Data collection integrated quantitative and qualitative methods to provide a comprehensive understanding of the program's impact.

4.3.1. Data Collection

Qualitative data were collected to explore changes in teacher attitudes and instructional practices throughout the CAR process. The ETs completed two questionnaires: a pre-questionnaire (26 items) capturing background information, teachers' views on geometry, teaching practices, perceptions of student difficulties, and familiarity with Van Hiele's and Duval's frameworks (Appendix 11, Antunović, 2024a), and a post-questionnaire (15 items) evaluating perceived impacts of the program on teaching practices and student learning (Appendix 12, Antunović, 2024a). Items employed Likert scales, Osgood semantic differentials, and open-ended questions questionnaire, aligned with previous research on attitudes in mathematics (e.g., di Martino, 2011, 2015; TIMSS & PIRLS, 2019) and validated by experts. Based on pre-questionnaire results, semi-structured interviews were conducted to deepen understanding of teacher perspectives. Typical questions included: "Why is teaching geometry complex and interesting for you?", "What activities or strategies do you use to overcome student difficulties?". Post-intervention interviews invited reflection on changes, e.g., "What changed in your practice after the DGT program?", "Which activities were most engaging for students?", and "Do you consider this form of action research valuable for professional development?". Each interview lasted ~30 minutes, was audio-recorded, and analysed thematically, with teacher dialogue subsequently used to illustrate key themes.

CTs completed a separate 28-item questionnaire overlapping partly with experimental surveys but also addressing their methods for teaching polygons and circles (Appendix 13, Antunović, 2024a), administered after their teaching.

Table 1 provides an overview of the questionnaire categories of their administration across ETS and CTs, summarizing the focus of each instrument before and after the DGT program.

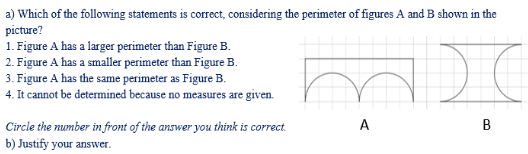
Table 1. *Overview of Questionnaire Categories Across Experimental and Control Groups Teachers*

Category	Before DGT Program (ET)	After DGT Program (ET)	After Teaching Polygons & Circles (CT)
Demographics	Gender, vocation, workplace, teaching experience, promotion status	– (same teachers, not repeated)	Same as ET
Teachers' Views of Geometry & Teaching	Importance, difficulty, contributions (reasoning, spatial, interdisciplinary), perceptions (complex/simple, etc.), guiding factors, preferred topics, challenges, self-evaluation	Changes in teaching, influencing factors, usefulness of program, motivation, applicability, reflection on student thinking (VH, GFA)	Similar to pre-test (importance, perceptions, guiding factors, challenges, self-evaluation)

Category	Before DGT Program (ET)	After DGT Program (ET)	After Teaching Polygons & Circles (CT)
Teachers' Perceptions of Students	Student difficulties, performance vs. math, performance by grade/topic, spatial-visual abilities, competencies (drawing, constructing, solving, proving), attitudes, % solving tasks	Changes in teacher–student relations, student abilities, outcomes (communication, problem-solving, proving, creativity), attitudes, % solving tasks	Similar to pre-test + achievements vs. math, factors for success/struggles, additional problems
Teaching Practices	Lesson planning, topic selection, instructional methods, classroom management, student assessment, reflection, adaptation to diverse abilities	Indirect: impact of program	Explicit: textbook use, ICT (GeoGebra, OneNote), manipulatives, inquiry/problem-solving, projects
Theories	Familiarity with VH and GFA	Usefulness of VH and GFA	Familiarity with VH and GFA

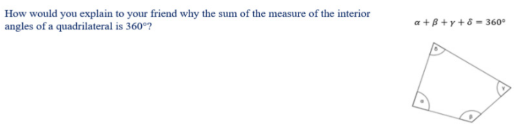
During the intervention, the researcher conducted observations of teacher–student interactions, activity design, and student engagement (Schoenfeld, 2013). These were complemented by field notes, student notebooks, photographs, and video recordings, documenting the implementation of the DGT program and providing evidence of changes in instructional practices and classroom dynamics.

Quantitative data were collected through pre- and post-tests designed to measure changes in students' geometric thinking, specifically the acquisition of Van Hiele (VH) levels and coordination of GFA types. The pre-test (Appendix 9, Antunović, 2024a), developed by the researcher, provided baseline data for the DGT program, while the post-test (Appendix 10, Antunović, 2024a) was collaboratively redesigned with the experimental teachers (ETs). To assess students' progress over the three-and-a-half-month intervention, VH sublevels (10/11, 20/21, 30/31, capturing partial versus complete acquisition) and GFA categories (2 = low interplay to 5 = high interplay, with 1 reserved for unsolved tasks) were used, allowing a nuanced evaluation of students' geometric thinking and development. Figure 2 presents examples of the test items, illustrating how the tasks were designed to capture both Van Hiele levels and GFA categories.



Task 4

Two figures on a square grid are given. The task investigates students' ability to compare the perimeters of the figures and to without relying on formulas, measurements, calculation, or estimation. The focus is on assessing students' understanding of the concept of perimeter, their ability to provide arguments, and their use of standard mathematical language.



Task 8

A picture is provided together with symbolic notation, and students are asked to explain or prove why the sum of the interior angles in a quadrilateral is 360°. The task examines whether students construct a proof empirically (through example), descriptively, or formally

Figure 2. *Examples of pre-test items*

4.3.2. Data Analysis

Quantitative and qualitative analyses were conducted on separate datasets to provide complementary insights into student outcomes and teacher practices. Quantitative analyses of pre- and post-test results were performed using SPSS, including Wilcoxon Signed-Rank, Mann-Whitney U, Kolmogorov-Smirnov tests, and Spearman correlations, focusing on Van Hiele levels and Duval's GFA.

Qualitative data — including pre- and post-questionnaires, interviews, classroom observations, and control group questionnaires — were analysed thematically. Key themes were identified across data sources, with teacher and student dialogue used to illustrate recurring patterns. Observations focused on teacher-student interactions, activity types, and levels of student engagement (Schoenfeld, 2013). Observations and supplementary materials (notes, photographs, videos) were triangulated with questionnaire and interview data to ensure that findings reflected consistent patterns rather than isolated opinions. Thematic analysis captured shifts in teacher attitudes, instructional strategies, professional growth, and their relationship to student engagement and understanding.

In this article, only selected quantitative and qualitative results are presented, focusing on summarized trends and representative examples that illustrate the impact of the DGT program on student learning and the relationship with changes in teacher attitudes and practices, while full datasets and detailed analyses are in Antunović (2024a).

4.4. Description of DGT program

The DGT program was designed as a collaborative action research project, grounded in van Hiele's model of geometric thinking and Duval's theory of GFA. The program focused on 7th-grade geometry topics — polygons and circles — chosen for their central role in fostering students' visual reasoning, construction skills, and conceptual understanding (MZO, 2019). The program unfolded through three iterative cycles that combined seven teacher–researcher workshops and eighteen classroom observations. Each cycle was structured to strengthen ETs involvement, bridge theory and practice, and transform textbook tasks into exploratory, dialogic, and student-centred activities (Duval, 2014). A variety of geometric tools (including GeoGebra, grids, and geoboards) and manipulatives supported students' visual, discursive, sequential, and operative apprehension of geometric figures (Duval, 1995, 2017). In contrast, the CTs followed the prescribed curriculum, without professional development related to Duval's and van Hiele's theory. Their teaching was not observed, and their contribution was limited to lesson reports and a post-teaching questionnaire.

The first cycle established the organizational and ethical groundwork for the intervention. Introductory meetings presented the research framework, clarified the teachers' roles, and defined the observation and testing plan. A collaborative climate was built, aligning the researcher's aims with the ETs pedagogical goals. During this phase, pre-tests were administered to both EG and CG students, while ETs completed a questionnaire and participated in semi-structured interviews to capture their initial perspectives.

The second cycle centred on theoretical grounding and instructional planning. Workshops introduced van Hiele's levels of geometric thinking and Duval's GFA, prompting ETs to reflect on how students perceive, interpret, and transform geometric figures. The researcher shared analyses of pre-test results, along with insights from ET interviews and questionnaires, highlighting common student difficulties such as weak visualization skills and challenges with dimensional deconstruction. Based on these findings, the program was refined. ETs received a curated selection of ~20 scientific papers on geometric thinking, action research, and non-measurement and open-ended tasks. These articles were linked to 7th-grade curriculum topics and included examples of visualization, cognitive apprehension, and problem-solving strategies. These resources helped ETs reflect on their instructional strategies, personalizing them to better fit their teaching contexts. Practical tools were introduced and discussed during workshops, with additional resources organized via Google Drive and video meetings. This collaborative design phase ensured that theoretical constructs were explicitly connected to teachers' classroom practices.

The third cycle emphasized implementation, co-creation, and reflection. It included two workshop modules — one on polygons, one on circles — in which the researcher presented materials structured around Duval's GFA with the analyses of the tasks from textbooks (Table 2).

Table 2. *Example of Textbook Analysis with Suggested Activities*

Topic	Textbook Content	Suggested Activity / Focus
Polygons	Definitions, examples, regular vs. irregular	Justify classifications, use counterexamples, apply dimensional deconstruction
Diagonals & Angles	Diagonals, angle sums	Explore multiple solution paths, connect formulas to visual proofs
Regular Polygons	Properties, constructions	Hands-on constructions, guided exploration
Perimeter and Area of Polygons	Standard perimeter/ area calculations	Project tasks on invariance of area, use of grids, comparing and transforming figures
Circle & Disk	Perimeter, area, arc length	Inquiry-based discovery of π, explore area of disk and its parts, connect polygons to circles, use GeoGebra

These materials guided teachers in designing exploratory tasks that coordinated different types of apprehension, supported visual reasoning, and fostered classroom dialogue. Teachers were encouraged to pose open-ended prompts that scaffolded student thinking and how to reformulate textbook tasks into inquiry questions. For example, when assigning a polygon construction task, adding the question "Describe the steps you used in your construction." prompted students to articulate their construction process, coordinating sequential apprehension and discursive apprehension and developing mathematical communication skills.

Workshops systematically addressed each GFA type and their coordination. Teachers learned to design non-numerical tasks (Duval, 2014; Tumova & Vondrova, 2017), apply construction tasks with descriptive instructions, and use different tools to encourage students in operative apprehension. In this way, exploration of geometric properties occurred without reliance on calculation, fostering a mathematical way of "seeing" figures (Duval, 2017). After each workshop, ETs implemented selected tasks in class, adapting them to their context and incorporating project-based assignments. Their role evolved gradually: from following researcher-prepared materials to actively designing tasks and lessons.

The researcher shifted from directing instruction to facilitating reflection and refinement through post-observation discussions. This cycle concluded with a collaboratively designed post-test, carefully aligned with the program's objectives. Across the implementation, 18 classroom observations were conducted, each followed by reflective dialogue between the researcher and ETs.

According to teacher lesson reports, the total number of lessons was balanced across experimental and control groups, ensuring that all students covered the required 7th-grade geometry content (MZO, 2019). Project tasks and supplementary topics varied by group, as shown in Table 3.

Table 3. *Overview of conducted lessons and content*

Group	Polygons (no. of lessons)	Circle (no. of lessons)	Total	Project Tasks & Additional Content
EG1	17	14	31	Project tasks (e.g., polygons in coordinate system; area of disk and circular segment). Additional topics: central & inscribed angle; Thales's theorem; chordal quadrilateral.
EG2	22	12	34	Project tasks (e.g., tiling the plane; non-numerical tasks). Additional topics: relations between two circles, line and circle.
CG1	16	23	39	No project tasks. Additional topics: central & inscribed angle; Thales's theorem; relations between circles/lines.
CG2	20	10	30	Project tasks (e.g., diagonals of polygons; perimeter of circle/arc length). No additional topics.

5. RESULTS

The results include both qualitative data from classroom observations, teacher questionnaires, and interviews conducted before and after the DGT program, and quantitative data from students pre- and post-tests assessing students' VH and GFA i.e. development of geometrical thinking.

5.5. Qualitative Results

The qualitative analysis integrates data from pre- and post-implementation questionnaires, interviews with ETs, classroom observations, and a questionnaire completed by CTs. Thematic analysis revealed four key themes, which are traced across pre-implementation data, classroom observations during the DGT program, and post-implementation findings:

- Challenges in teaching geometry – including teachers' limited familiarity with Van Hiele and Duval frameworks, reliance on concrete measures, and perceived difficulty of geometry content.
- Shifts in instructional strategies – such as the adoption of hands-on, multi-representational, collaborative, and inquiry-based approaches.
- Changes in teacher attitudes and professional motivation – reflecting increased confidence, willingness to experiment, and appreciation for theory-informed instruction.
- Student engagement and learning outcomes – illustrated by higher participation, deeper reasoning, and improved understanding of fundamental geometric concepts.

5.5.1. Pre-Implementation Findings

Analysis of pre-implementation questionnaires and interviews with the experimental teachers (ET1 and ET2) highlighted several challenges in teaching geometry. Both teachers considered geometry as important as other areas of mathematics but recognized it as more demanding to teach. They emphasized its contribution to reasoning, spatial abilities, and interdisciplinary thinking, describing it as complex, interesting, and dynamic. However, both teachers reported limited familiarity with Van Hiele's and Duval's theoretical framework, indicating a gap between their practical strategies and theoretical knowledge. ET1 demonstrated awareness of diverse student needs, emphasizing differentiation to engage all learners: ET1: "I always start from the simplest one… those who find it difficult don't feel bad, and those who find it very interesting don't get bored."ET2 struggled with abstract topics such as congruence and similarity, considering them too advanced for younger students and suggesting postponing them to higher grades: ET2: "These contents are complex for students; they seem not mature for this content."

Both teachers observed that students relied heavily on concrete measurements rather than reasoning conceptually. Questionnaire data confirmed these observations: ET1 rated her students' performance and her teaching effectiveness more positively, while ET2 reported lower student achievement and interest. Across both teachers, students were generally stronger in drawing and constructing than in reasoning or proofs. These findings, observed across all lessons and teacher responses, indicated the need for structured professional development to support theory-informed geometry instruction.

5.5.2. Observations During DGT Program Implementation

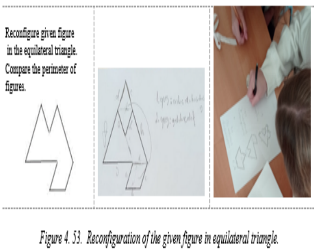
Classroom observations of 18 lessons (Table 4) revealed that the DGT program fostered theory-informed teaching strategies.

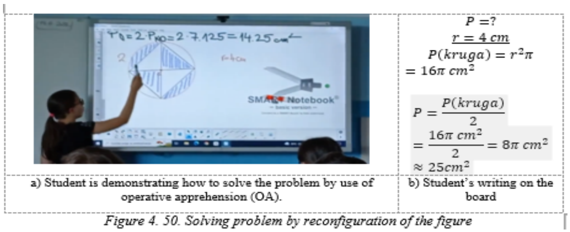
Table 4. *Overview of the lesson's observations*

EG1 (9 lessons)	EG2 (9 lessons)
Polygons in coordinate system; area (project task)	Tiling the plane
Central and inscribed angle	Perimeter and area of polygons
Tasks on central & inscribed angles	Perimeter and area of polygons (extension)
Thales's theorem; triangle → chordal quadrilateral	Perimeter of a disc
Perimeter of a disc; circular arc	Area of a circle
Perimeter of a disc; circular arc (continued)	Arc length & area of a circular segment
Perimeter of disc – modelling	Perimeter & area of disc and its parts

EG1 (9 lessons)	EG2 (9 lessons)
Area of a disc	Connecting perimeter/area of disc & polygons
Area of a circular section	Relations: two circles, line & circle

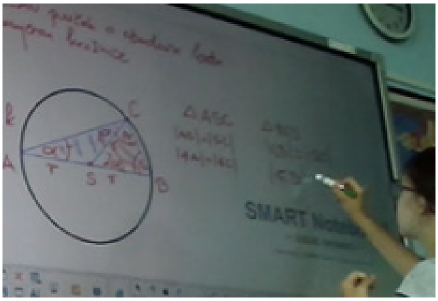
ETs consistently encouraged students to translate between verbal, visual, and symbolic representations, supporting deeper conceptual understanding. Activities such as reconfiguring figures, tiling the plane with polygons, and calculating areas of discs emphasized hands-on exploration and real-world connections, aligning with Duval's GFA framework (See Figure 3). Teachers facilitated collaborative discussions, prompting students to articulate strategies, justify solutions, and use precise terminology, thereby promoting independent reasoning and problem-solving. Observations confirmed that ETs successfully integrated multiple representations, concrete models, and structured communication, engaging students actively with geometric concepts.



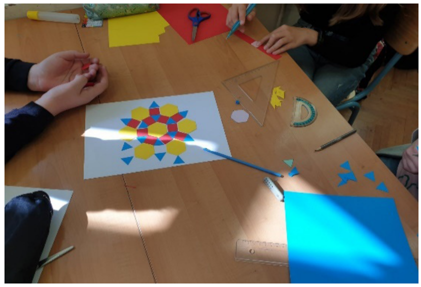


Activities of reconfiguring

Activity of determining the area of the disk



Activity of Thales's theorem of inscribed angle



Activity – Tilling the plane

Figure 3. *Teaching activities designed by the teacher of experimental group*

The teacher-student dialogue was well-implemented, consistently promoting reasoning, metacognition, and connections between visual, symbolic, and verbal representations. Teacher prompts such as *"What do you notice?", "Why is this angle 90°?",* and *"Can you prove that?"* guided students through structured reasoning rather than providing direct answers.

For example, in a lesson on Thales' theorem of inscribed angle, the teacher led students' step by step: labelling angles, applying the external angle theorem, identifying isosceles triangles, and deducing relationships, ultimately concluding that an inscribed angle on a diameter is a right angle. Prompts like *"What can you say now from this figure?"* and *"Can you explain in your own words?"* encouraged students to articulate relationships, use formal vocabulary, and engage in logical deduction, effectively supporting dynamic assessment.

Similarly, during lessons on the area of a disc, students explored geometric properties through visualization and manipulation:

ET: Great. And what do you notice when the number of sectors increases?

ES: The figure starts to look like a trapezium or parallelogram.

ES: Or a rectangle.

ET: Excellent. Now, imagine we construct an inscribed regular polygon with 20 sides, or 35, or even 360 sides, and then reconfigure the disc. Can you visualize this?

ES: Yes.

ET: What geometric figure does this new figure resemble in your mind?

In lessons on tiling the plane, students reasoned about geometric constraints: "Tile the plane with at least two regular polygons and explain why it works, discussing your ideas with peers." They identified that polygons must have congruent sides and that hub-vertex angles must sum to 360° for proper tiling.

Across all lessons, teacher prompts scaffolded reasoning, encouraged articulation of geometric relationships, and fostered connections across multiple representations. This dialogue-based approach replaced rote instruction with active engagement, reinforcing conceptual understanding and problem-solving skills.

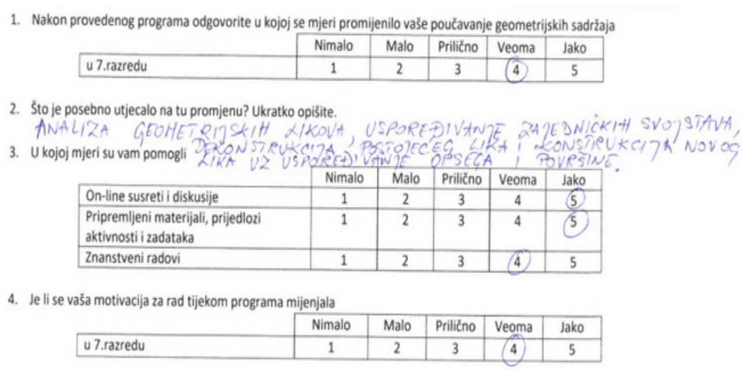
5.5.3. Post-Implementation Findings

Post-program questionnaires and interviews indicated substantial changes in teaching practices, professional motivation, and student engagement. Both teachers shifted from rigid, calculation-focused tasks to exploratory, inquiry-based approaches. ET1 described giving students opportunities to discover geometric relationships through practical activities: ET1: "I now give tasks so students can explore the basic figures in different situations." ET2 highlighted tasks that allowed students to manipulate shapes without relying on concrete measurements: ET2: "I have even done a lot of these tasks without concrete measures… and I think that's great."

ETs observed increased student engagement, particularly during collaborative group work, and reported that students developed a better understanding of area, perimeter, and spatial relationships.

Further, they emphasized the balance between theory and practice. While van Hiele's and Duval's frameworks were valuable, practical classroom applications and hands-on workshops had the most immediate impact on instructional change (See Figure 4). Both ETs expressed intent to continue using new strategies and considered action research a useful approach for addressing broader challenges in teaching mathematics.

These post-implementation findings were consistent across both experimental teachers and triangulated with classroom observation data.



1. Nakon provedenog programa odgovorite u kojoj se mjeri promijenilo vaše poučavanje geometrijskih sadržaja

	Nimalo	Malo	Prilično	Veoma	Jako
u 7.razredu	1	2	3	4	5

2. Što je posebno utjecalo na tu promjenu? Ukratko opišite.
ANALIZA GEOMETRIJSKIH LIKOVA, USPOREĐIVANJE ZAJEDNIČKIH SVOJSTAVA, DEKONSTRUKCIJA POSTOJEĆEG LIKA I KONSTRUKCIJA NOVOG LIKA UZ USPOREĐIVANJE OPSEGA I POVRŠINE.

3. U kojoj mjeri su vam pomogli

	Nimalo	Malo	Prilično	Veoma	Jako
On-line susreti i diskusije	1	2	3	4	5
Pripremljeni materijali, prijedlozi aktivnosti i zadataka	1	2	3	4	5
Znanstveni radovi	1	2	3	4	5

4. Je li se vaša motivacija za rad tijekom programa mijenjala

	Nimalo	Malo	Prilično	Veoma	Jako
u 7.razredu	1	2	3	4	5

1. After completing the program, to what extent has your teaching of geometric content in the 7th grade changed? 4
2. What specifically influenced that change? Briefly describe. - *Analysis of geometric figures, comparison of common properties, deconstruction of existing and construction of new figures, along with comparison of perimeter and area.*
3. To what extent did it help you: Online meetings and discussions - 5, Prepared materials-5, suggestions for activities and tasks Scientific papers - 4.
4. Did your motivation for working in the 7th grade change during the program? 4

Figure 4. *A part of the answers of post questionnaire of ET2*

5.5.4. Control Group Teacher Responses

Control group teachers (CT1 and CT2) completed questionnaires after teaching polygons and circles (Appendix 13, Antunović, 2024a). Their responses revealed reliance on traditional, textbook-driven methods with limited use of hands-on or collaborative strategies. Neither teacher was familiar with Van Hiele or Duval frameworks, and student engagement remained low. CT1 emphasized mathematical content and processes, while CT2 focused on student motivation but struggled with limited materials and time. These observations, collected across the control group, contrasted sharply with the experimental teachers' adoption of interactive, theory-informed practices.

In summary, the thematic analysis, integrating questionnaires, interviews, and classroom observations, shows that the DGT program had a transformative effect on ETs. Prior to the program, ETs relied on intuition, struggled with abstract concepts, and lacked theoretical grounding. During implementation, they employed hands-on, multi-representational, and collaborative strategies informed by Van Hiele and Duval frameworks. Post-program data indicate enhanced professional motivation, improved student engagement, and a commitment to sustaining innovative practices. In contrast, CTs maintained lesson driven approaches, with limited innovation and persistent challenges in student engagement. By examining the data from all lessons and teachers, including questionnaires and interviews, we identified recurring patterns that provide a clear picture of the program's overall impact. Overall, the DGT program successfully integrated theoretical frameworks with practical strategies, fostering meaningful teacher growth and improved student learning outcomes.

5.6. Quantitative results

Quantitative data were collected through pre-, and post-tests administered to students, based on Duval's GFA framework and van Hiele's levels of geometric thinking. Only students who completed both tests were included in the analysis, resulting in a final sample size of 78. Statistical analyses revealed a significant improvement in the performance of the experimental group compared to the control group, highlighting the effectiveness of the DGT program (see Table 5).

The pre-test results showed no statistically significant differences between the experimental and control groups, indicating that the groups were comparable in their initial performance. For VH the p-value was p=0.094, and for GFA, the p-value was p=0.107. Most students in both groups were positioned between the first and second van Hiele levels, corresponding to the “Recognition” and “Analysis” stages (sublevel 20). Regarding GFA, the majority of students fell into the intermediate category (category 3), reflecting a perceptual approach to geometric figure. This suggests deficiencies in visualization skills and a limited understanding of foundational geometric concepts.

The post-test results demonstrated a marked improvement in the experimental group. A significant number of these students progressed to higher van Hiele levels, with many attaining levels 3 (sublevels 30 and 31), indicating an ability to relate geometric properties and engage in informal deductive reasoning. Regarding GFA, students in the experimental group developed a more sophisticated, mathematically way of “seeing” geometric figures. They demonstrated strong coordination of Duval’s types of apprehension, particularly excelling in operative and discursive apprehension. In contrast, the control group exhibited no statistically significant changes from pre-test to post-test in either van Hiele levels or GFA. Median scores remained stable, suggesting that teaching methods of teachers of control group were ineffective in enhancing students geometric thinking, coordination of different GFA or problem-solving skills. Wilcoxon test results indicated significant within-group improvements for the experimental group in both VH (p=0.030) and GFA (p=0.020). However, no significant changes were observed in the control group (p=0.108 for VH and p=0.593 for GFA). Mann-Whitney tests confirmed significant between-group differences in post-test scores for VH (p=0.022) and GFA (p=0.020).

Table 5. *Change of Students' Van Hiele Levels and Interplay of Geometrical Figure Apprehensions – Results of Pre and Post-Test*

Groups	VH Levels Pre-test Median (Range)	VH Levels Post-test Median (Range)	Within-group p*	GFA Pre-test Median (Range)	GFA Post-test Median (Range)	Within-group p*
Experimental (N=40)	20 (10–30)	21 (10–31)	0.030	3 (2–5)	4 (2–5)	0.020
Control (N=38)	20 (10–30)	20 (11–30)	0.108	3 (2–4)	3 (2–4)	0.593
Between groups – p **	0.094	0.022	–	0.107	0.020	–

*Wilcoxon test for within-group comparison
**Mann-Whitney test for between-group comparison

These results underscore the effectiveness of the DGT program in developing and fostering higher levels of geometric thinking and enhancing students' capacity to analyse and reason about geometric figures. The experimental group's success reflects the impact of different pedagogical strategies, such as collaborative and hands-on learning experiences, which were integral to the program.

6. DISCUSSION AND CONCLUSION

The findings highlight how CAR can transform teacher attitudes and instructional practices in geometry education, leading to improved teaching effectiveness and student outcomes.

The first research question asked: *How does the CAR process influence teachers' attitudes toward geometry and their instructional practices?* The presented data revealed that CAR had a profound impact on teachers' attitudes. ETs reported increased confidence and enthusiasm for teaching geometry after participating in the DGT program. This change aligns with Hannula's (2002) framework, which highlights how positive affective responses, such as enjoyment and reduced anxiety, encourage teachers to engage more actively with their subject. ET1's more significant shift further supports the idea that attitudes can change through structured, reflective interventions (Reid, 2015). ETs began to view geometry as an accessible and engaging subject rather than an abstract challenge, reflecting the findings of Russo et al. (2020), who noted the correlation between teacher enthusiasm and improved classroom dynamics. Results from observation also showed improvements in teaching practices among ETs. Inquiry-based, student-centred approaches became more prevalent, leading to improved student outcomes. The collaborative nature of CAR, involving shared experiences and continuous reflection, was pivotal in fostering this change. As a core component of CAR, iterative cycles of planning, action, observation, and reflection encouraged teachers to refine their practices, echoing the findings of Bruce et al. (2011) on the transformative potential of collaboration in action research. These results align with broader literature emphasizing the critical role of professional development in reshaping teacher attitudes and practices (e.g., Guskey, 2002; Yuan & Lee, 2015).

The second research question asked: *In what ways do changes in teachers' attitudes and teaching methods, as a result of CAR, impact students' understanding of geometric concepts and their problem-solving abilities?* Result of student performance, based on the portion of quantitative data presented in this paper alongside qualitative classroom observations, suggests a positive association between teacher practices and student learning trends. Students in the experimental group demonstrated improvements in overall test performance, indicating enhanced understanding of geometric concepts and problem-solving skills. Observational data revealed active student engagement, with learners collaboratively exploring problems and reasoning through solutions during hands-on, inquiry-based activities. These patterns are consistent with Duval's and van Hiele's frameworks, which highlights the role of visualization, reasoning, and problem-solving in geometry education. The adoption of hands-on, inquiry-based teaching strategies by experimental group teachers appeared to support these developments. Teachers guided students through Duval's stages, including dimensional deconstruction and coordination of different types of apprehension, promoting deeper conceptual understanding (Duval, 2017). These observations align with findings on the benefits of inquiry-based teaching for developing problem-solving skills (Vula, 2013; Betts et al., 2017; Schoenfeld, 2019). By contrast, the control group, which employed primarily lecture-based methods, showed no significant improvement. While only partial quantitative data are presented here, these results underscore the potential value of integrating innovative, theory-informed teaching strategies to foster engagement and support student learning, consistent with prior research (Reid & Ali, 2015).

The findings of this study suggest that CAR, facilitated through the DGT program, provides a promising model for improving teacher attitudes and student learning outcomes in geometry. The iterative, reflective nature of CAR empowered teachers to adapt their instructional methods, leading to more dynamic classroom environments and better student performance. These findings align with Schoenfeld's (2019) advocacy for bridging research and practice, demonstrating how evidence-based strategies can inform effective teaching. Furthermore, the dissemination of findings through workshops and conferences allowed experimental teachers to share their experiences, further highlighting the potential of CAR to inspire broader changes in instructional practices. As Wright (2021) notes, such collaborative efforts not only enhance professional development but also create ripple effects, encouraging innovation across educational settings. Furthermore, these findings support existing research (e.g. Guskey, 2002; Burt et al, 2020) emphasizing the importance of teacher attitudes and innovative instructional methods in improving student engagement and achievement.

The findings also underline the importance of tailoring professional development programs such as the DGT initiative to the specific needs of teachers and students. As highlighted in the intervention, customized resources, targeted support for addressing diverse student abilities, and alignment of content with developmental readiness all contributed to the effectiveness of the program. These implications suggest that professional development should not only transmit theory but actively adapt to classroom realities, echoing the significance model presented in this study. Moreover, by integrating CAR with frameworks like van Hiele's levels, Duval's GFA, and Schoenfeld's TRU Math, the program demonstrated how multiple theoretical perspectives can work together to produce meaningful transformations in teacher practice and student learning.

In conclusion, the CAR model implemented through the DGT program provides a valuable framework for professional development in geometry education. By focusing on reflective practice and collaborative learning, the program aimed to implement key and recent research findings on geometric thinking, enabling teachers to apply these findings in their teaching methods. As Schoenfeld (2019) suggests, the productive synergy between research and practice is crucial for driving meaningful change in education. Through fostering a culture of continuous learning and collaboration, CAR has the potential to drive lasting improvements in teaching and learning, ultimately enhancing educational outcomes on a broader scale. The process enriches both students and teachers, nurturing a supportive and enthusiastic teaching environment. More importantly, it fosters trust not only between the teacher and students but also between the researcher and the teacher. This mutual trust and collaboration are essential in creating a dynamic, thriving educational environment. Just as the teacher-student relationship in the classroom is vital for student success, the researcher-teacher relationship is equally important for the growth and effectiveness of educational practices. Through CAR, this "triangle" of researcher-teacher-student is strengthened, leading to a positive cycle where both teaching and learning are continuously enhanced.

Future research should examine how the DGT program and similar CAR-based approaches can be scaled across diverse educational contexts — including schools with varied socio-economic profiles and teaching traditions — and explore their potential to transform teaching practices in geometry and other mathematical disciplines.

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DEVELOPMENT OF PRE-SERVICE TEACHERS' MATHEMATICAL LANGUAGE THROUGH GEOMETRICAL CONSTRUCTIONS PROBLEMS

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Abstract

Geometric constructions with a straightedge and compass (whether with paper and pencil or with appropriate software) are widely recognized as a valuable teaching tool in developing visualization skills, promoting a deeper understanding of geometric concepts, and strengthening the development of mathematical thinking and reasoning. In the construction process, the key part is the way of processing the geometrical figure which is created or reconstructed for the purpose of solving a problem or proving a statement. An important aspect of processing a geometrical figure is recognizing the elements of the figure that are mathematically important and connecting them into the appropriate functional sequence of steps as well as describing the execution steps (shorter: sequential apprehension). In this paper, the focus is on the development of sequential apprehension, and in particular on the development of the mathematical language of pre-service primary education teachers during solving geometric problems that involve constructing or reconstructing complex geometrical figures. This analysis suggests that sequential apprehension of geometrical figures, as well as mathematical language, do not develop automatically, through the mere execution and description of practical activities, but rather through the gradual structuring of the construction process and describing it. In this context, the use of different representations of the same situation (visual representations, language descriptions, symbolic notations) and the establishment of functional connections among them are especially important.

Keywords: *language description, problem-solving process, sequential apprehension, symbolic notation, visual representation*

1. INTRODUCTION

Euclidean geometry is a fundamental field of mathematics, but learning it presents several challenges, including understanding and defining geometric concepts, formulating and proving geometric statements, and posing and solving geometric problems (Fujita & Jones, 2002; Kondratieva, 2013; Yevdokimov, 2024). Mastering geometric concepts and establishing connections between them requires a range of skills, such as visualizing abstract concepts, describing observed relationships with precise language, and using symbolic notation (Baranović, 2024).

Geometric constructions are a crucial aspect of geometry and have been recognized as valuable tools for both learning and teaching the subject. According to Fujita et al. (2010), geometric constructions serve as an effective means of bridging the practical process of their derivation with the theoretical process of constructing the corresponding proof. Moreover, the value of geometric constructions in enhancing content knowledge, fostering logical reasoning, and promoting argumentation has been confirmed (Yevdokimov, 2024). Geometric constructions also play a significant role in increasing student activity, motivation, and fostering a positive attitude toward learning geometry (Massarwe, 2023). Despite these benefits, the use of geometric constructions in mathematics education has declined in recent decades, with tools like compasses and rulers often being used solely for drawing basic geometric figures (Courtney & Armstrong, 2021; Kondratieva, 2013; Yevdokimov, 2024).

The situation in Croatian and Slovenian curricula (KNPM, 2019; Žakelj et al., 2008; Žakelj et al., 2011) and primary school mathematics textbooks mirrors that of many other countries: geometric constructions are presented to a lesser extent, typically in procedural terms, with little focus on explanations (either verbal or symbolic), argumentation, or proof.

One possible reason for the decreased use of geometric constructions in mathematics teaching could be the challenges associated with managing them in the classroom, particularly when it comes to fostering logical reasoning, argumentation, and proof (Yevdokimov, 2024). Therefore, it is crucial to enhance teachers' knowledge and skills, preparing them to effectively address these challenges. Since mathematical language is a key aspect of managing geometric constructions, our research focuses on developing mathematical language through the solving of geometric construction problems.

2. THEORETICAL BACKGROUND

In this paper, a geometric construction refers to the visual representation of a geometric figure created solely using a straightedge and compass (either with paper and pencil or appropriate software). The straightedge is a tool for drawing straight lines without a measuring scale (hereafter referred to as a ruler), while the compass is used for drawing circles and circular arcs, as well as for measuring distances.

In the context of learning and teaching, it is important to understand the different types of geometric constructions and their gradation. According to Klančar et al. (2019), constructions can be classified into basic, elementary, and complex categories. Basic constructions involve the use of a ruler (to draw a straight line, ray, or line segment) or a compass (to draw a circle or circular arc). Elementary constructions are more complex then basic ones, involve the use of both a ruler and compass, and can be broken down into a short sequence of basic constructions. Common elementary constructions encompass the construction of congruent segments and angles, perpendicular bisectors of line segment, midpoints, angle bisectors, lines perpendicular or parallel to a given line, dividing a segment into equal parts or a given ratio, and the four basic triangle constructions. Complex constructions refer to all other constructions that can be broken down into a finite sequence of elementary constructions.

According to Polya (1966), geometric construction is a unique type of problem that serves as both a "task to find" and a "task to prove." It is a "task to find" because it requires constructing a geometric figure that satisfies all the given elements and conditions, with the active use of instruments such as a ruler and compass. These tasks demand practical skills and an explanation of the construction steps. At the same time, geometric construction is a "task to prove" because it necessitates verifying the correctness of each construction step and providing valid justification through appropriate mathematical argumentation. Both aspects of geometric construction thus require the use of precise mathematical language. Teaching and learning geometric construction problems is challenging for both students and teachers due to the dual nature of geometric concepts (Fishbein & Nachlieli, 1998) and the difficulties in working with geometric figures (Yerushalmy & Chazan, 1990). For instance, various studies (Chin et al., 2022; Retnawati et al., 2017; Ulusoy, 2016) have highlighted that students often struggle with understanding the properties and constructions of parallel and perpendicular lines. This basic geometric knowledge, however, is essential for comprehending and constructing more complex geometric figures, such as rectangles (Chin et al., 2022). Furthermore, Duval (1995) points out that many students face challenges in perceiving different dimensions and fully apprehending geometric figures.

For effective work with geometric figures, Duval (1995) proposes a theoretical framework that distinguishes four types of understanding: perceptual, sequential, discursive, and operational. According to Duval, the processing of a figure begins with a purely perceptual recognition of what the figure shows — this is referred to as perceptual apprehension (PA). PA occurs unconsciously and is necessary when working with geometric figures, but it is not sufficient for fully understanding them. Following PA, a conscious effort is required to uncover the mathematical meaning of the figure within its context, which is achieved through a combination of sequential, discursive, and operational processing. To master the transition from pure perceptual apprehension to recognizing the mathematical significance of a figure, students need to develop various visualization processes, first separately and then in coordination (Duval, 1995). This paper focuses on the development of sequential apprehension (SA), specifically on how the mathematical language of pre-service primary education teachers evolves while solving geometric construction problems. SA refers to the process of constructing or reconstructing geometric figures and describing these processes in terms of critical steps. It also involves recognizing the mathematically important elements of a figure and connecting them into a functional sequence of execution, along with describing the execution steps. Additionally, the construction process requires identifying and understanding the critical steps involved in creating a geometric figure, while reconstruction involves discovering these steps according to the mathematical properties of the constructed figure. Critical steps encompass all the basic and elementary constructions into which complex structures can be decomposed. In short, sequential understanding depends on the technical aspects of the construction process and the mathematical properties inherent in basic and elementary constructions.

In the teaching and learning process, it is crucial to utilize various representations of the same problem situation and to develop the skill of flexible transitioning between visual representation, linguistic description, and symbolic notation, without allowing any one to dominate (referred to as the VLS system). Working with multiple representations is neither a linear nor straightforward process, but it can be mastered through focused learning and teaching (Baranović, 2024). Geometric construction problems provide a natural environment for developing mathematical language within the VLS system.

3. RESEARCH DESIGN, AIMS AND QUESTIONS

This research is part of a broader study focused on the development of mathematical language, thinking, and reasoning. It involves designing appropriate tool-based tasks, observing the teaching-learning process, facilitating classroom discussions, conducting qualitative analysis of student work, and administering task-based interviews. The aim of this research was to explore effective methods for teaching geometric constructions to foster a sequential apprehension of geometric figures. Specifically, the research sought to investigate how to design tool-based tasks that support the development of the mathematical language necessary for advancing mathematical thinking and reasoning. To guide this exploration, the following research questions were posed:

- How and to what extent pre-service primary education teachers develop SA and mathematical language while solving problems involving geometric constructions?
- What are the typical obstacles and mistakes encountered by pre-service primary education teachers in the problem-solving process involving geometric constructions?

4. METHODOLOGY

This research is situated within an interpretative paradigm and adopts a qualitative approach (Cohen et al., 2007). The collected data were analyzed through a combination of qualitative analysis of pre-service primary education teachers' work, task-based interviews, and documentation from classroom discussions.

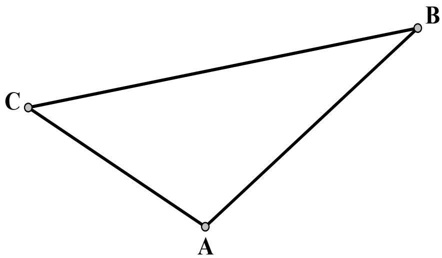
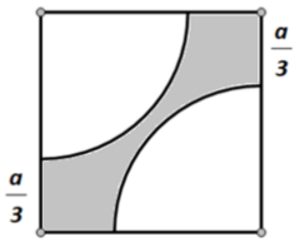
4.1. Sample

This research involved all pre-service primary education teachers at the University of Split, Croatia, who attended the geometry course during the summer semesters of 2023 and 2024. A total of 90 pre-service teachers participated, with 45 in 2023 and 45 in 2024. Participation was voluntary. The Euclidean geometry course is offered in the second year of the undergraduate program, with an average student age of 20.

4.2. Instrument

The instrument consists of two geometric tasks (Task 1 and Task 2), each requiring both construction and description (see Table 1) and participants were given 15 minutes to complete each task.

Table 1. *Geometric construction problems*

In the triangle *ABC* from the picture, construct altitudes v_a, v_b i v_c, the orthocenter *O* and the inscribed circle *k(S, r)*. Describe the construction process step by step.	A square of side length *a* and a colored part inside it, determined by circular arcs as in the picture, are given. Construct the given image for $a = 5$ *cm*. Describe the construction steps.
	
Task 1	Task 2

Since the construction of a figure depends on the technical aspects of the construction process and the mathematical properties inherent in basic and elementary constructions (Duval, 1995), through the construction of given figures we sought to examine students' conceptual knowledge and construction skills, as well as ways of describing them. The tasks are designed to match the content covered in real-time geometry classes and to be of appropriate complexity, i.e. according to the number and complexity of elementary constructions into which a figure can be decomposed. For this reason, the first task covers concepts related to a triangle, and the second task covers concepts related to a quadrilateral.

Both tasks use basic constructions that are assembled into a complex construction. Although the tasks are not comparable at first glance, they test basic knowledge of geometry, which is understood as knowledge and understanding of basic concepts in geometry and solving simple problems. Since both tasks require the implementation and description of the construction process and the use of appropriate symbolism, the progress of the participants can only be observed over a period of time. During this period, the participant is present in the learning process and builds on their knowledge.

4.3. Data collection and analysis

Data were collected during regular mathematics classes in the summer semester. In the first part of the semester, participants completed Task 1 after studying the characteristic points of a triangle, and in the second part, they completed Task 2 after studying quadrilaterals. Our analysis focused on participants who completed both tasks, examining three key elements for each: understanding of concepts, the skill of constructing elementary constructions and decomposition methods, as well as a description of the construction process.

In Task 1, we analyzed the understanding of three geometric concepts (altitude, orthocenter, and inscribed circle), the skills required to construct two elementary constructions (perpendicular through a point to a line and the angle bisector), and the decomposition method for complex constructions, along with the language of description. In Task 2, we analyzed the understanding of two geometric concepts (square and circular arc), the skills needed to construct three elementary constructions (congruent line segments, perpendicular through a point to a line, and dividing a segment into equal parts), and the decomposition method for complex constructions, along with the language of description. After reviewing all participants' papers, 50 remained who participated in writing both tasks. In addition to general observations based on cross-sectional qualitative and descriptive analysis, this paper provides a detailed qualitative analysis of two tasks performed by two participants with different scores (Student A and Student B) on both tasks to assess whether progress was made.

5. RESULTS AND DISCUSSION

Figure 1 presents Student A's solutions to both tasks, followed by a transcript. Analyzing the visual representation of the construction in Task 1 (see Transcript from the Figure 1, SP1 to SP4), it is evident that the participant correctly constructed the perpendicular from the vertex of the triangle to the opposite side, and the orthocenter as the intersection of the constructed lines, but did not construct the altitude, i.e. he did not highlight the altitudes as segments on the constructed perpendiculars. Namely, the altitude of a triangle is defined as the line segment from the vertex of the triangle to the base of the perpendicular to the line of the opposite side (Gutierrez & Jaime; 1999). Furthermore, Student A also correctly constructed the center of the inscribed circle, but the circle's radius is not constructed or drawn at all. By definition, a circle is considered constructed when both the center and radius are constructed, not just the center.

When analyzing the description of the construction in Task 1, it is clear that the participant decomposed the complex task into smaller steps, though not into elementary constructions. The description uses a descriptive language with limited mathematical vocabulary and no symbolic notations.

By considering both the visual representation and the description together, it becomes apparent that the participant misunderstands the concept of the altitude (treating it as a line) and consequently incorrectly defines the orthocenter as the point of the intersection of the altitudes instead as the intersection of the altitude lines. It is important to consider the lines of the opposite sides and the lines of the altitudes because in an obtuse triangle, the two altitudes fall outside the triangle, so the altitudes do not intersect each other, but the lines of the altitudes do intersect (Gutierrez & Jaime; 1999). Additionally, the participant shows a misunderstanding of how to construct a circle, as they neither determined the radius constructively nor descriptively.

Analyzing Student A's solution to Task 2 (see Transcript from the Figure 1, SP5 to SP6), it initially appears that no progress has been made, as the student provided a drawing and a brief description without symbolic notation instead of a full construction. However, the description reveals two key components, including some elementary constructions. Although it is unclear why a complete construction and detailed description were not provided, it is possible to reconstruct the intended solution from the written description. In this form, a mathematical vocabulary and more precise description can be developed more easily.

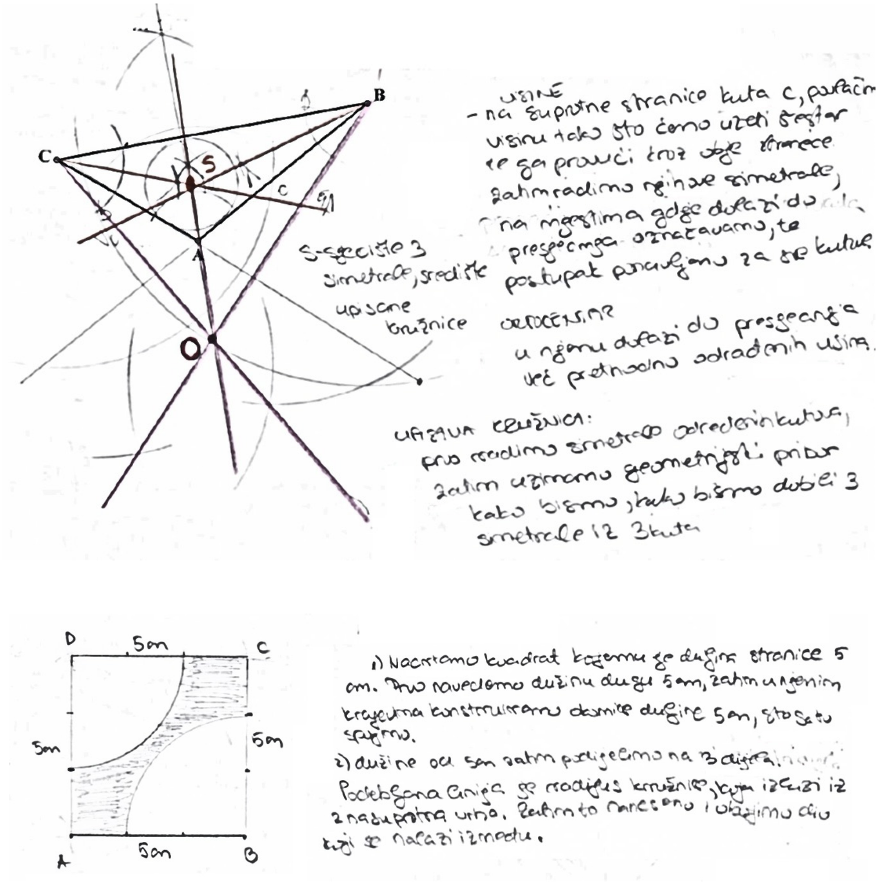


Figure 1. *Student A's solution of Task 1 and Task 2.*

Transcript from the Figure 1 with the abbreviation SP for a separate part that encompasses a meaningful whole (description, conclusion, symbolic notation, etc.), from the top to the bottom:

SP1: ALTITUDES: I draw altitudes to the opposite sides of angle *C* by taking a compass and passing it through both sides. Then we draw their perpendicular bisectors, marking the points where they intersect and repeating the process for all angles. [*The figure shows the extension of the sides* $\overline{AB}$ *and* $\overline{AC}$*, and the correct construction of the perpendiculars through the vertices of the triangles A, B and C to the lines of opposite sides*.]

SP2: ORTHOCENTER: In it, the previously worked altitudes are intersected. [*The figure shows a point highlighted in the intersection of a ray drawn from the vertices of a triangle, labelled O*.]

SP3: INSCRIBED CIRCLE: First we make the bisectors of certain angles, then we use geometry tools to get 3 bisectors from the angle. [*The picture shows the correct construction of the bisectors of all angles of a triangle, as rays from vertices A, B and C.*]

SP4: S - intersection of 3 bisectors, center of inscribed circle. [*The figure shows a point marked at the intersection of the described rays and labelled S. The figure also shows a circle with center S, touching the sides of the triangle from the inside*.]

SP5: We draw a square with a side length of 5 cm. First we specify a segment with length of 5 cm, then we construct perpendiculars of 5 cm length at its ends, and then we connect them. [*The figure shows only a sketch of square ABCD (not a construction) with marked lengths of 5 cm on all sides*.]

SP6: Then we divide the 5 cm long sides into 3 parts. The bold line is the radius of the circle that comes out of the two opposite vertices. Then we apply that and get the part that is in between. [*The figure shows the division marks on each side of the square, the bold part of the side* $\overline{AB}$ *for the radius of the circle, and the corresponding circular arcs*.]

It is already evident from this example (Figure 1) that mathematical language develops quite slowly, especially that the transition to symbolic language is not so simple.

Figures 2 present Student B's solutions to Tasks 1, followed by a transcript. In Figure 2, the visual representation shows that Student B's construction is nearly identical to Student A's from Figure 1, but the description is more comprehensive (see Transcript from the Figure 2, SP1 to SP10). In the description, Student B breaks down the complex construction into elementary constructions, briefly outlines the critical steps, and introduces symbolic notation to a limited extent. However, when both the visual representation and the description are considered together, the same misconceptions about altitude, orthocenter and circle as in Student A's work are evident. In addition, from the symbolic notation (see Transcript from the Figure 2, SP10), it is evident that the participant is thinking about the radius of the circle, but incorrectly identifies it: instead of looking at the distance from the center to the side of the triangle, he takes the distance from the center to the vertex of the triangle, so there is no need to construct this additionally.

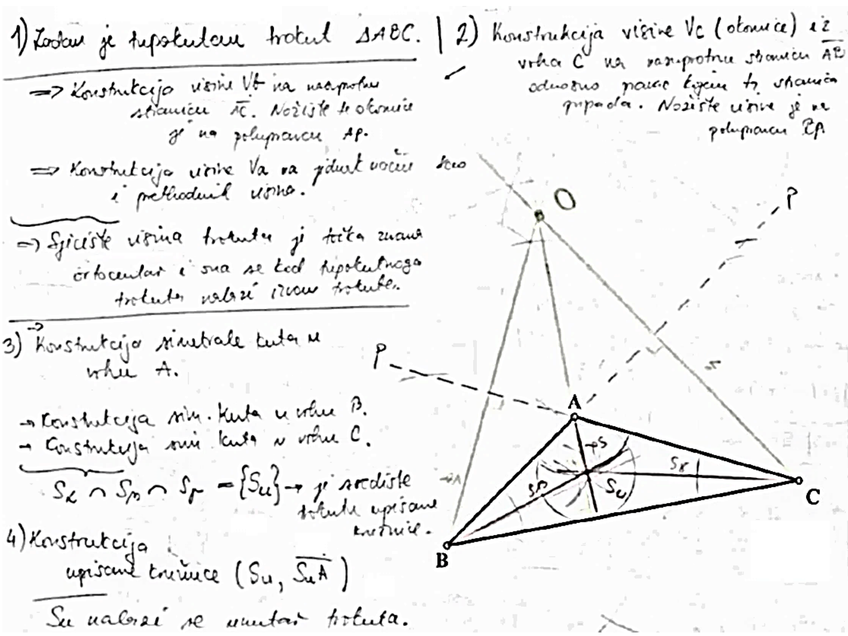


Figure 2. *Student B's solution of Task 1*

Transcript from the Figure 2 with the abbreviation SP for a separate part that encompasses a meaningful whole (description, conclusion, symbolic notation, etc.), from the top to the bottom:

SP1: An obtuse triangle ΔABC is given.

SP2: $\Rightarrow$ Construction of the altitude v_b to the opposite side $\overline{AC}$. [*The figure shows a ray from the vertex B, the label* v_b *on it, and the label for a right angle at the base of the perpendicular.*] The base of this perpendicular is on the ray A_p. [*The figure shows an extension of side* $\overline{AC}$ *and the label p on it.*]

SP3: $\Rightarrow$ Construction of the altitude v_a in the same way as the previous altitude. [*The figure shows a line through the vertex A, the label* v_a *on it, and the label for a right angle at the base of the perpendicular*.]

SP4: $\Rightarrow$ Construction of the altitude v_c (perpendicular) from vertex *C* to the opposite side $\overline{AB}$, that is, to the line to which that side belongs. [*The figure shows a ray from the vertex C, the label* v_c *on it, and the label for a right angle at the base of the perpendicular.*] The base of the altitude is on the ray B_p. [*The figure shows an extension of side* $\overline{BA}$*, and the label p on it.*] (This part is separated on the right, and the arrow $\rightarrow$ indicates insertion into the section before the conclusion.)

SP5: $\Rightarrow$ The intersection of the altitudes of a triangle is a point called the orthocenter, and in an obtuse triangle it is located outside the triangle. [*The figure shows a point marked at the intersection of the described rays and labelled O.*]

SP6: $\rightarrow$ Construction of the angle bisector at the vertex of triangle *A*. [*The figure shows the corresponding circular arcs, the ray from vertex A, and the label* s_α *on it.*]

SP7: $\rightarrow$ Construction of the angle bisector at the vertex of triangle *B*. [*The figure shows the corresponding circular arcs, the ray from vertex B, and the label* s_β *on it.*]

SP8: $\rightarrow$ Construction of the angle bisector at the vertex of triangle C. [*The figure shows the corresponding circular arcs, the ray from vertex C, and the label* s_γ *on it.*]

SP9: $s_\alpha \cap s_\beta \cap s_\gamma = \{S_U\}$ $\rightarrow$ The center of the inscribed circle of a triangle. [*The figure shows a point marked at the intersection of the described rays and labelled* S_U.]

SP10: Construction of an inscribed circle $\left(S_U, \overline{S_U A}\right)$. S_U is located inside the triangle. [*The figure shows a circle with center* S_U*, touching the sides of the triangle from the inside.*]

From this example it is clear that symbolic notation develops gradually, by replacing parts within the linguistic description. Also, the precision of symbolic notation does not develop all at once.

Figures 3 present Student B's solutions to Tasks 2, followed by a transcript. After additional study and practice, Student B made notable progress: his construction became more precise, and the description more concise, with increased use of symbolic notation (see Figure 3 and Transcript from the Figure 3, SP1 to SP8). It is evident from Student B's construction that he correctly reconstructed the visual representation, accurately constructed the square, and drew the circular arcs within it. He also correctly divided the sides of the square into three equal parts; however, instead of constructing the division, he drew a parallel line. This aligns with the findings of Chin et al. (2022) regarding the challenges students face in constructing parallel lines.

According to the description (see Transcript from Figure 3, SP5, SP7), the participant provides some descriptions only symbolically, while other parts remain concise linguistic descriptions. This further confirms the observation that the transition from linguistic description to symbolic notation is a very slow process, and for the purpose of deeper understanding, both should be developed mutually with the support of visual representation (Baranović, 2024).

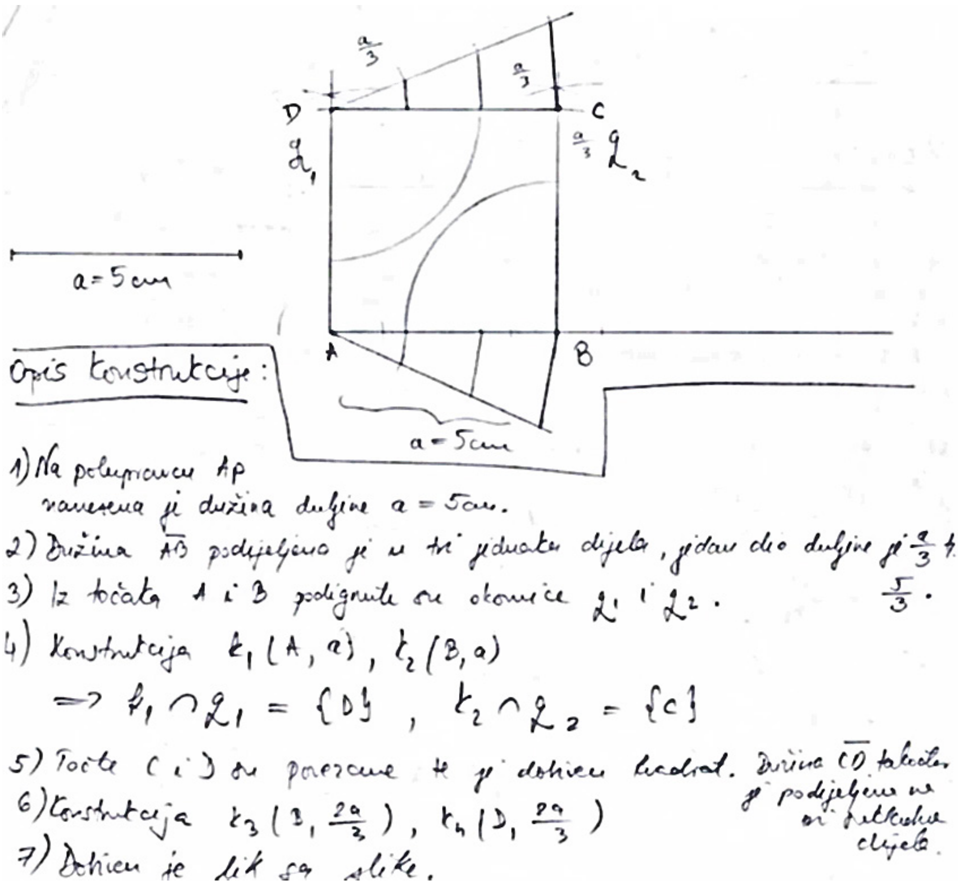


Figure 3. *Student B's solution of Task 2.*

Transcript from the Figure 3, from the top to the bottom:

SP1: Construction description: [*The figure shows the construction above the line.*]

SP2: A line segment of length $a = 5cm$ is transferred on the ray A_p. [*The figure shows a segment line on the left, a length a = 5cm indicated below it, a ray drawn to the right from point A, and a point B marked on the ray as the other end point of the transferred segment.*]

SP3: The length $\overline{AB}$ is divided into three equal parts, one part of the length is $\frac{a}{3}$, i.e. $\frac{5}{3}$. [*The figure shows an auxiliary ray drawn from A, below the starting ray, with three points on it that define three equal parts, counting from point A, the connection (segment) of the last point and point B, parallels to that connection through the highlighted points, and finally two points on segment* $\overline{AB}$ *at the intersection of the parallels and the starting ray.*]

SP4: From points *A* and *B*, perpendiculars q_1 and q_2 are raised. [*The figure shows the corresponding circular arcs, perpendicular rays from points A and B, and the labels* q_1 *and* q_2 *on them, respectively*.]

SP5: Construction $k_1(A,a)$, $k_2(B,a)$; $\Rightarrow k_1 \cap q_1 = \{D\}$, $k_2 \cap q_2 = \{C\}$. [*The figure shows the corresponding circular arcs, highlighted points on the perpendiculars* q_1 *and* q_2, *and named with D and C, respectively.*]

SP6: Points *C* and *D* are connected and a square is obtained. The segment line $\overline{CD}$ is also divided into three congruent parts. [*The figure shows segment line* $\overline{CD}$ *and its division into three congruent parts in the same way as in step SP3*.]

SP7: Construction $k_3\left(B,\frac{2a}{3}\right)$, $k_4\left(D,\frac{2a}{3}\right)$. [*The figure shows the corresponding circular arcs*.]

SP8: The figure from the picture is obtained.

It is also evident from the Figure 3 that the division into equal parts was made on two sides of the square, and the reason for this was clarified through a task-based interview. Below is an excerpt from the interview (abbreviations: T for Teacher and S for Student):

T: Is it necessary to construct the division into equal parts of the two sides of the square?

S: Well... it's not. The radii of the circles are equal.

T: Then, why did you construct in that way?

S: To see the division on both sides.

T: How would you have acted if you had not divided the other side?

S: I would take 2/3 of the side in the opening of the compass and draw the circular arcs from B and D.

This example clearly shows that the participant did not initially consider how to reach a solution more efficiently. However, during the conversation, he realizes that the procedure could have been simplified.

Based on the data collected from classroom discussions, participants' written papers, and task-based interviews, several general characteristics can be observed. Initially, participants, who are not yet proficient with the tools, focus more on procedures than on concepts, aligning with the findings of Massarwe (2023) in his experimental study. This lack of conceptual focus is also evident in their descriptions, which are initially unstructured, overly descriptive, and lacking in mathematical vocabulary, often with little to no symbolic notation. However, after a period of study and practice, students gradually begin to recognize the critical steps of a construction, learn how to break down complex constructions into simpler, elementary and basic components, expand their mathematical vocabulary, and incorporate more extensive symbolic notation.

Typical mistakes made by participants during the construction and description process can be observed. In Task 1, participants to a large extent treat the altitude as a line rather than a segment, both in construction and description, leading to the incorrect conclusion that the orthocenter is the intersection of all altitudes. The altitude as a line segment is emphasized by slightly more than a fifth of participants (11 out of 50; 22%), but less than a tenth of the participants (4 out of 50; 8%) explain that the orthocenter is formed as an intersection of altitude lines. This confirms the results of other researchers that "the concept of altitude of a triangle is not an easily grasped concept by either pupils or preservice teachers" (Gutierrez, A. & Jaime, 1999, p.272), which may be a consequence of using prototypical triangle figures (e.g. Hershkowitz, 1989). Additionally, participants to a large extent (38 out of 50, 76%) correctly construct only the center of an inscribed circle, neglecting to determine the radius, while only two of them say how to find the radius.

In Task 2, more than half of the participants (28 out of 50; 56%) construct a 90° angle and incorrectly label it as a perpendicular construction, while only a little more than a tenth of the participants (7 out of 50, 14%) correctly construct the perpendicular. High share of participants (32 out of 50; 64%) divide two sides of the square into equal parts instead of just one, which, while not incorrect, is unnecessary. These observations further support the findings of Chin et al. (2022) on the difficulties of constructing perpendicular and parallel lines.

Typical obstacles that hinder the solution of geometric construction problems of this type can also be identified. The precision of the construction is primarily affected by poor tool handling (such as the ruler and compass), while the correctness of the construction is undermined by incorrect, incomplete, or misunderstood definitions of terms, as well as a lack of understanding of the properties involved in defining a figure (e.g., a circle is determined by its center and one of its points). Additionally, the accuracy of the construction is compromised by inadequate skills in breaking down a given figure into critical construction steps – specifically, decomposing it into a structured sequence of elementary constructions and then linking the elements of the figure to the underlying concepts, which represents a higher level of construction knowledge.

The results presented above demonstrate that mathematical language can be effectively developed through solving geometric construction problems. This progress was supported by the use of multiple representations of the same situation within the VLS system, along with executing the construction in accordance with critical steps. These strategies provided a clear structure and fostered the development of SA, which is essential for solving this type of problem (Duval, 1995). These findings align with the results from experimental research on the effectiveness of the VLS system in learning and teaching geometry (Baranović, 2024).

6. CONCLUSION

The paper presents a qualitative research focused on enhancing mathematical language through solving geometric construction problems. The findings highlight geometric constructions as an effective context for developing mathematical language, facilitated by the functional integration of visual, linguistic, and symbolic representations of the same situation. Additionally, the results suggest that the sequential figures apprehension is effectively fostered through a structured decomposition of the construction into critical steps. The research also demonstrates that mathematical vocabulary, precision, symbolic notation, and conceptual understanding develop progressively as students describe and execute these structured steps.

Further research could explore the impact of these activities among in-service teachers and their students, offering insights on how to integrate them into teaching materials. Additionally, it would be valuable to investigate whether improving mathematical language in this way enhances students' ability to argue and prove the validity of constructions.

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ON THE DEVELOPMENT OF PROBLEM-BASED TEACHING RESOURCES FOR THE FOSTERING OF MATHEMATICALLY INTERESTED PRIMARY SCHOOL STUDENTS

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Abstract

The development of problem-based teaching resources for students with limited mathematical experience is a challenge. How much openness is possible and how much guidance is necessary to enable productive, potentially successful mathematical activity? The article reports on exemplary experiences that we were able to gather in a long-term fostering project for mathematically gifted and interested primary school students.

Keywords: *mathematical giftedness, problem solving, problem-based teaching resources, openess, guidingness*

1. THE *MATH INVESTIGATORS* – A FOSTERING PROJECT

"Talents in themselves are always only possibilities …" (Stern, 1916, p. 110) This old but still valid statement emphasizes the importance of fostering for mathematically gifted students, which starts early, is implemented continuously and is conceptually appropriate (regarding objectives, content and methods). It also highlights the responsibility of teachers and maths didacticians. Of course, fostering must begin in regular math lessons. Nevertheless, extracurricular fostering programs for high-achieving or gifted students are also important.

For more than 10 years, the university fostering project *Math Investigators* has been aimed at mathematically gifted third and fourth graders, but is also open to 'only' mathematically interested and high-achieving children.

Based on Stern's concept, we understand mathematical giftedness as potential for exceptional achievement in the field of mathematics, which, however, does not automatically manifest itself in outstanding (school) performance. This makes identifying mathematical giftedness difficult and uncertain. Mathematical giftedness can be described in particular by special cognitive abilities. Krutetzkii (1976) presented a corresponding catalog for older students as early as the 1970s, and Käpnick (1998) or Assmus (2018) later did so for primary school age children. The extent of such abilities should, if possible, be explored through long-term observation and analysis of the mathematical activities of the respective student.

On the other hand, high performance in elementary school math classes does not automatically imply mathematical giftedness, among other reasons because instruction is often heavily focused on calculation and mastery of corresponding procedures, while more challenging content (e.g., mathematically rich patterns and structures, problem solving, argumentation) plays a less prominent role.

We therefore distinguish between mathematically gifted and high-achieving students, although students can, of course, belong to both groups. Because gifts can only emerge and develop over the course of a student's school career, our program is aimed at both groups.

With this project, we combine various aims. First and foremost, our focus is on fostering gifted or high-achieving primary school students. In addition, the project is integrated as a part of a project seminar into the university curriculum for pre-service primary school teachers who have chosen math as their main subject. The project seminar consists of two parts. At the beginning of each semester, the first part, which is theory-based, focuses on fundamental theories and models relating to giftedness, mathematical giftedness, and the fostering of mathematically gifted students. The second, practice-oriented part of the seminar involves students accompanying and organizing student meetings in the fostering project. The teacher students plan the meetings with the primary school students, elaborate teaching materials (supported by the authors) and finally evaluate them. Of course, we support the teacher students by providing the idea for the problems, but the specific questions or tasks and the design of the worksheets are left to them.

The project is of course also an important opportunity for research, for example on mathematical giftedness, problem solving or mathematical creativity. With regard to fostering materials, it also provides the opportunity to conduct design-based research. These goals and our efforts to achieve them are of course not independent of underlying conceptions as visualized in Figure 1.

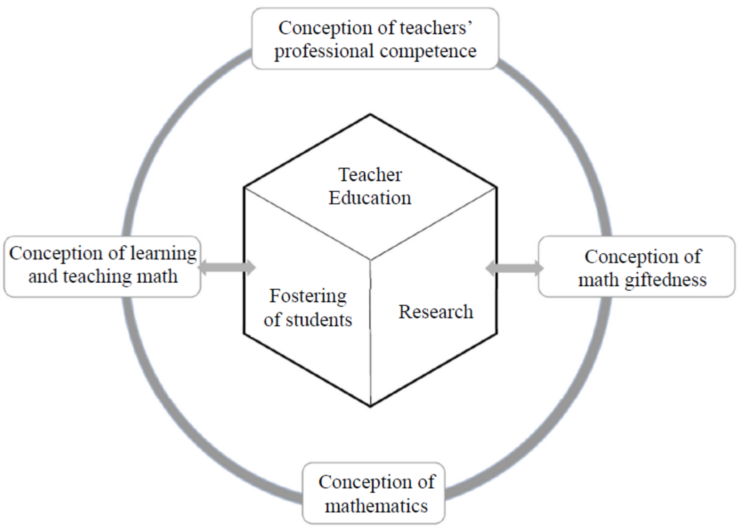


Figure 1: *The Math Investigator project*

2. PROBLEM BASED TEACHING RESOURCES

There is not enough space here to discuss possible concepts for the long-term fostering of mathematically high-achieving and gifted students in detail. Internationally, there is widespread agreement concerning the content of fostering programs that an enrichment approach (in relation to school math) should be implemented. If mathematics is seen as an open, dynamically developing system and a creative product of research communities, a problem-orientated approach is an obvious basic *methodological* orientation of fostering. Therefore, the long-term goal of a fostering program could be to introduce students to mathematical theory-building processes. Theory-building processes often begin with the exploration of a mathematically comprehensive situation from which the initial problem is drawn. Working on the problem, variations and extensions can then prompt further motives for working – a cycle of problem solving and the development of new or advanced problems is initiated. From the resulting outcomes, developed strategies and aids, from generated terms and discovered correlations, a "web of experience and knowledge" is shaped which finally has to be optimized (e.g., regarding elegance, fitting to existing theories, generalisability), preserved and integrated into already existing knowledge (e.g. Fritzlar, 2008). As a first step in this direction, for the fostering of less experienced (primary school) students not only isolated mathematical problems, but also several mathematically interconnected problems could be used, known as problem fields (e.g. Pehkonen, 1991). Problems or problem fields should fulfil as many of the following requirements as possible:

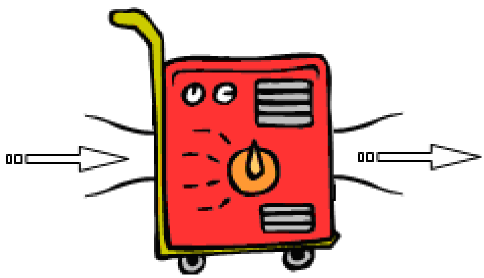
- The mathematical richness of the problems should be challenging even for high-achieving students without pre-empting the content of regular math lessons.
- Nevertheless, they should allow an easy entry and intermediate successes to maintain the students' motivation.
- There should be different ways of dealing with the problems.
- Problems should focus on aspects that challenge the special

abilities of mathematically gifted students as, e.g., working on patterns and structures, possibilities for justifying, generalizing, reverse thinking, changing representations, using heuristic elements, … (cf. e.g. Assmus, 2018).

- Problems should be characterized by increasingly open questions, variability and extensibility and thus also
- Provide opportunities for creative mathematical activities.

3. THE *NUMBER CONVERTER*

One example in which several of these requirements are fulfilled is the *number converter*.

Inputput	Processing	Output
6		3
5		4
9		8
8		4
11		10
1		0
12		6

The number converter works according to the following rules:

- If you enter an even number, the number is halved.
- If you enter an odd number, the number is reduced by 1.

Once you have converted a number, you can re-enter the new number into the number converter. You can do this until you reach 0. For example, if you start with the number 6, you get the sequence 6→3→2→1→0.

6 is a "Four-step-number", because you need four steps to reach 0.

You can now ask different questions: e.g. Find four-step numbers. How many exist? How many *n*-step numbers are to be found? Which is the largest number in each case?

Figure 2: *The Number Converter (Fritzlar, Rodeck & Käpnick, 2006)*

Most students find it easy to access the problem and can quickly generate their own examples. Additionally, there are various possibilities for finding specific *n*-step numbers, such as the four- or five-step numbers:

- Select any number, run the algorithm, assign to the number of steps
- Systematic checking of all numbers from 1 onwards
- Working backwards

The following table provides an overview of the *n*-step numbers and their quantity.

Steps	Numbers	Largest number	Quantity
1	1	1	1
2	2	2	1
3	3, 4	4	2
4	5, 6, 8	8	3
5	7, 9, 10, 12, 16	16	5
6	11, 13, 14, 17, 18, 20, 24, 32	32	8
7	15, 19, 21, 22, 25, 26, 28, 33, 34, 36, 40, 48, 64	64	13

Table 1: *n-step numbers*

The sequence of powers of two for the largest n-step numbers and the Fibonacci sequence for the quantities of the n-step numbers can be discovered. There are also “smaller” possible findings as “each sequence of steps ends with 2 → 1 → 0” or “the sequences of smaller *n* are contained in those of larger *n*” etc.

In order to discover such regularities, it is necessary to explore the problem extensively and to collect and organize many examples. As such exploration processes can vary greatly from individual to individual, on the one hand we want to give the students as much freedom as possible in our fostering project. On the other hand, they still have little experience of carrying out an extensive processing of mathematical problems independently and in a planned manner while keeping an eye on the objectives. This raises the question of how we can support them in organizing their working process without restricting their individual freedom too much.

4. *OPENESS VS. GUIDINGNESS*

This question is exemplary for a particular challenge in developing problem-based teaching resources, especially for young students with little mathematical experience**:** the balance between openness and guidingness. On the one hand, it is an important goal to enable students to do math as independently as possible. On the other hand, experience shows that primary school students often need support in a challenging math situation,

- to find initial examples that encourage further exploration,
- to find questions that are mathematically interesting and at the same time basically accessible for primary students,
- in the systematic collection of examples and data, and
- in developing suitable representations to facilitate the construction of patterns and the recognition of relationships.

In order to develop problem-based teaching resources that are as open as possible and that provide no more guidance than is absolutely required, it is necessary to make specific decisions for the individual learning group, which are often uncertain. Enabling pre-service teachers to experience this and sensitizing them to the balance between openness and guidingness are therefore important goals of our project seminars.

In our fostering project, we work with two groups of primary school students for grades 3 and 4 respectively. This gives us the opportunity to use problems twice and to revise the worksheets and lesson plans developed by the teacher students together with them between the two meetings.

The following two versions of questions and tasks originate from such a 'pair of meetings':

Version 1:	Version 2:
a) Here is space for your examples. What do you notice? b) Find all 4-step numbers! Which one is the largest? Which one is the smallest? How many 4-step-numbers exist? c) Find all 6-step numbers! Which is the largest? Which is the smallest? How many 6-step-numbers exist? d) Find all ___-step numbers! Which is the largest? Which is the smallest? How many exist?	a) Find all 4-step-numbers! Which one is the largest? Which one is the smallest? How many 4-step-numbers exist? b) Find all 6-step numbers! Which is the largest? Which is the smallest? How many 6-step-numbers exist? c) Find numbers with more than 4 steps! Find all! How many exist? Which is the largest? Which is the smallest? ___-step numbers ___-step numbers ___-step numbers

Figure 3: *Two versions of questions and tasks*

The two lesson plans differ firstly in that the first version started with a free exploration of step numbers without an explicit task. In some cases, the students invested a lot of time in these explorations, which was later lacking in the more advanced tasks. This did not contribute to a systematic exploration of the step numbers. In addition, many students were not very motivated to work on subsequent tasks, as they had already 'done a lot' in the beginning. In this group of students, in which version 1 was used, this open-ended task obviously did not encourage systematic enquiry and thus the seeking of regularities.

Secondly, another difference between the two versions can be seen in the last task. While in the first version only any additional step number was to be explored without limiting the number of steps *n*, in version 2 the task was to examine three different step numbers with more than four steps. It turned out that exploring only one additional step number did not contribute to recognizing patterns or relationships. Some of the students chose very small numbers (as in Figure 4) in order to minimize the amount of work involved. Others chose very large numbers for which they were unable to find all possibilities, resulting in the explorations not being helpful for recognizing patterns.

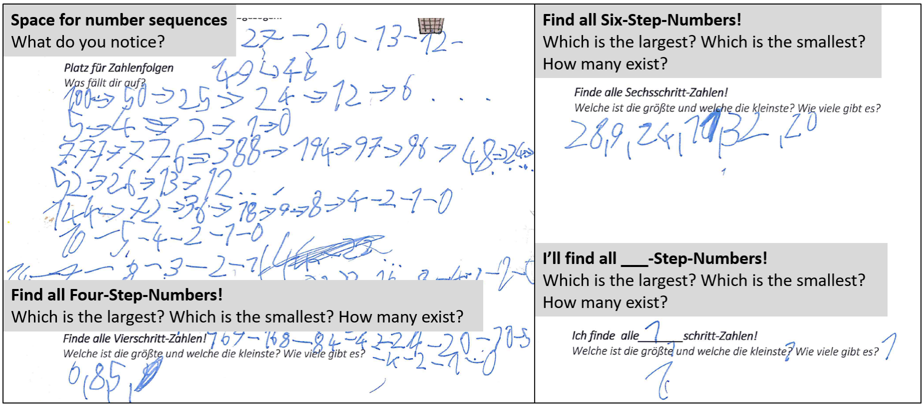


Figure 4: *Notations by students*

In the second version of questions and tasks (Figure 3), significantly more time was invested in the systematic exploration of selected step numbers. Two examples can be seen in Figure 5. Due to the free choice of numbers, however, the respective students (with one exception) did not consider any relationships between numbers. However, this could be done later in the plenum, where the results were summarized together and presented in a table. The free choice of numbers also meant that some students chose numbers for which they could not find all possibilities due to their size. It can therefore also be stated here that the free choice of numbers did not support the recognition of regularities in the involved group of students.

It can be summarized that also the other two tasks of the first version proved to be inappropriate in their succession. The exploration of four-step and six-step numbers does not encourage students to consider relationships between these step numbers and, for example, to generate six-step numbers based on four-step numbers. The discussion of the five-step numbers after the four-step numbers would be more suitable here.

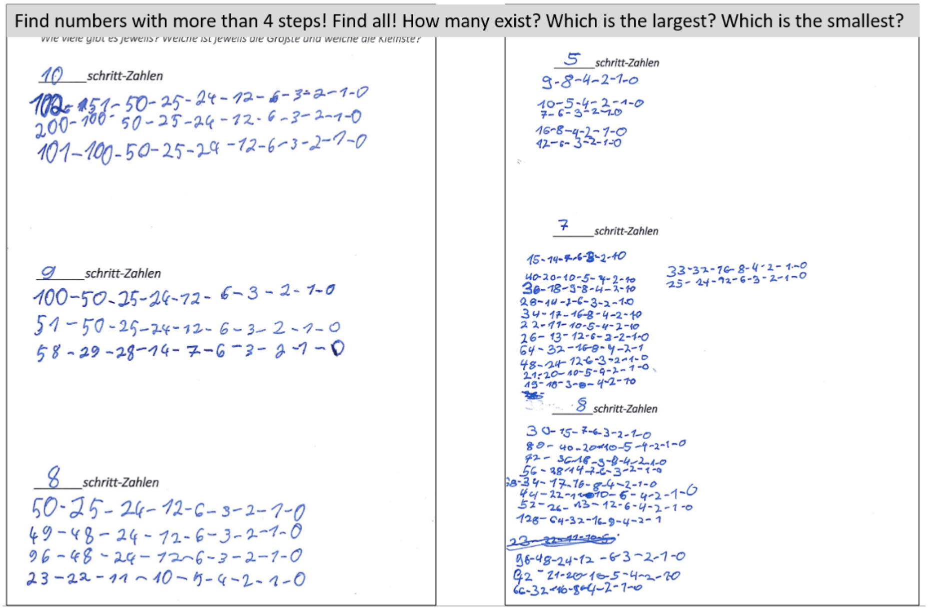


Figure 5: *Further notations by students*

It is not only the aspect for which *n* the students are explicitly asked to search for *n*-step numbers that can have a major influence on working processes and results. Instructions for notating the (intermediate) results can also have a decisive impact. In the fourth grade, after exploring four- and five-step numbers, the following task was tested once without and once with the specification of a table.

Answer the questions for other "step numbers" as well (one-step numbers, two-step numbers, three-step numbers, six-step numbers, seven-step numbers etc.). What mathematical relationships can you find?

Aaron worked without a table. Starting with the 0-step number, he wrote down all step numbers up to 5-step numbers, presumably systematically checking the numbers from 1 to 9 with regard to their number of steps. He then changed his course of action and decided to find a 20-step number. He started with 0 and was able to use the reverse relations. The illustration suggests that the calculation of the large numbers was error-prone and, above all, very time-consuming; there was no time left for a systematic search for n-step numbers for larger n. In addition, Aaron has not noted the quantity of n-step numbers for any n.

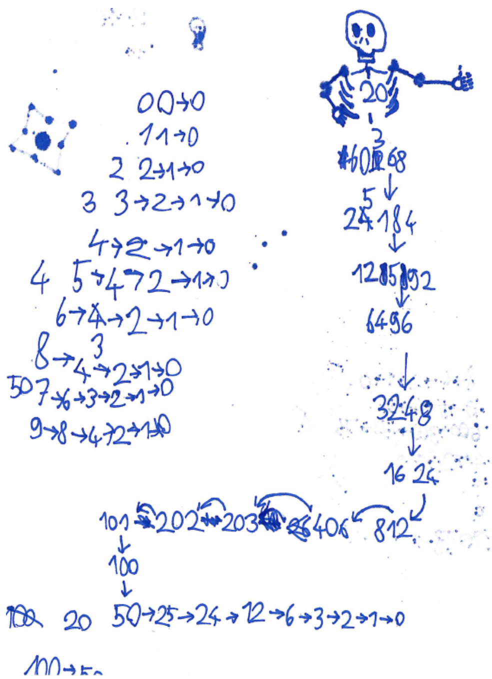


Figure 6: *Aaron's worksheet*

Bruno tried to write down the n-step numbers he had found in an orderly manner and to organise his notes graphically. In doing so, he has followed the instructions exactly in the sense that the previously investigated four- and five-step numbers are not included in the chart. However, this makes it very difficult for him to recognise patterns.

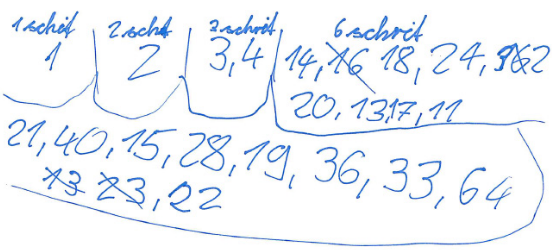


Figure 7: *Bruno's notes*

Henriette (Figure 8) independently created a systematic, quasi tabular notation for the *n*-step numbers she found, although the respective quantities are not noted. The coloring of the numbers is particularly interesting, as the girl uses it to make clear which numbers are produced by doubling or halving. Henriette used this relationship she had found to identify *n*-step numbers for larger *n* easily and without time-consuming verifications. However, it is not possible to find all *n*-step-numbers in this way; for example, in the eighth and ninth lines there are only even numbers.

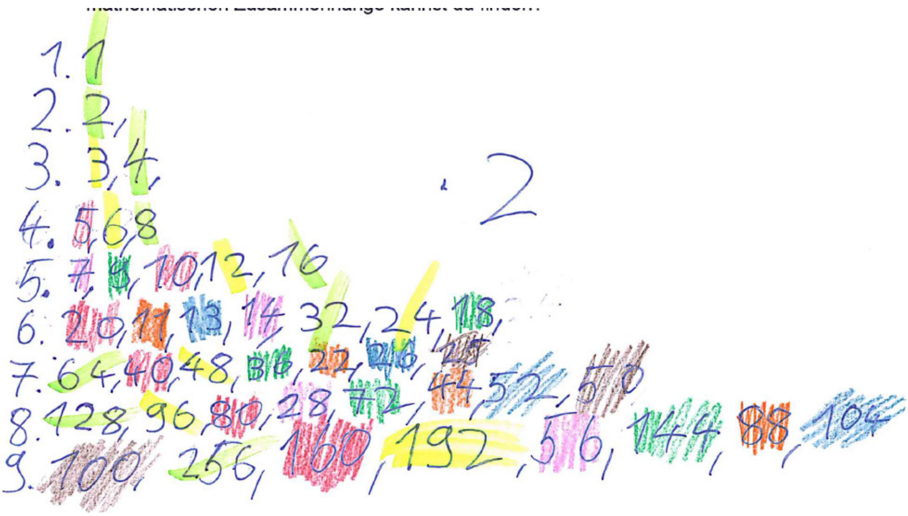


Figure 8: *Henriette's worksheet*

Sebastian was provided with the following table on a worksheet (Figure 9). Not only can it help with the organized notation of the various *n*-step numbers, the two columns on the right also draw attention to the respective quantity of *n*-step numbers and their largest. This helps students to find relations such as the doublings in the third column. But of course, a table provides no guarantees, as the entries in the far-right column show.

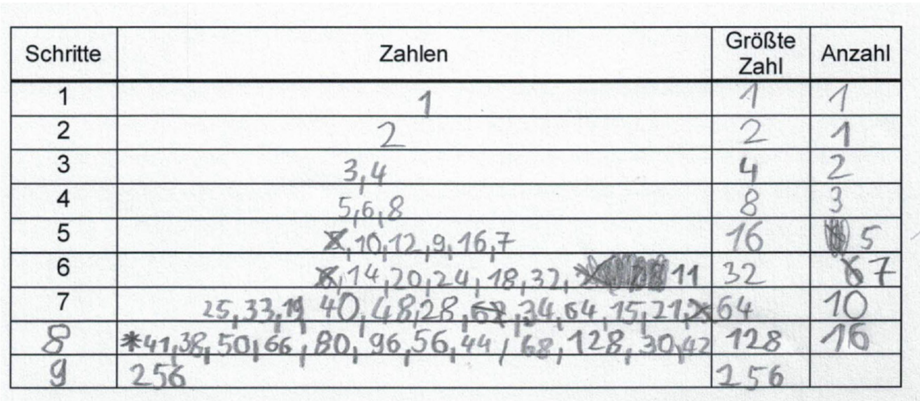


Figure 9: *Sebastian's table*

Walter also works with a table and succeeds in correctly completing the two columns on the right. In the notation below the table, the order of the numbers indicates that Walter also uses doubling to find *n*+1-step-numbers based on the *n*-step-numbers. In addition, even *n*-step-numbers are increased by 1. This enables him to find all *n*+1-steps-numbers. Perhaps it is also due to the formative power of the table that Walter, after he has found all *n*-steps-numbers, transfers them to the corresponding table row in the order of their size. However, Walter probably does not perceive the table as a limitation, as he naturally continues his patterns beyond the specified table rows.

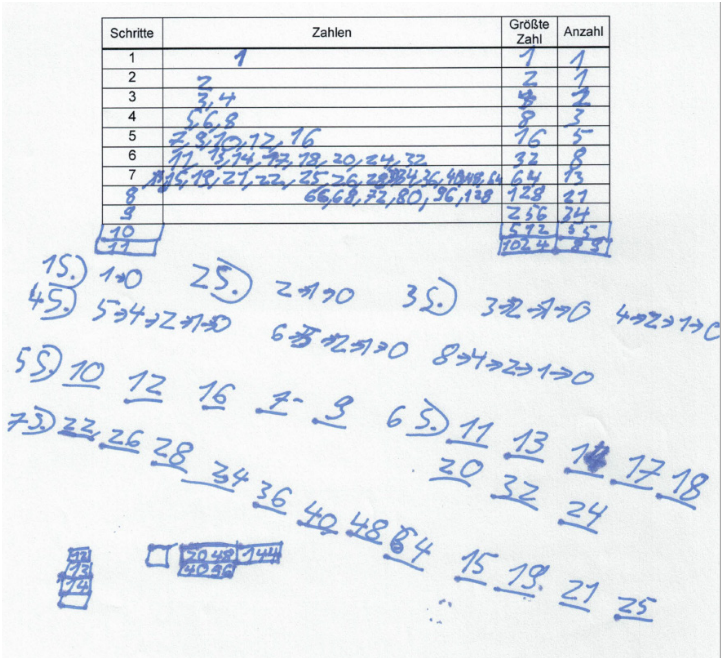


Figure 10: *Walter's worksheet*

The examples show: If no table was specified, students were given more freedom, which they used, for example by considering large or subjectively significant numbers. Also from a diagnostic perspective, it is very interesting to see which forms of representation the students found. However, without a table, there was often neither a systematic investigation nor a focus on particularly relevant aspects. Based on this, it can be concluded that not specifying a table did not support the recognition of regularities in the involved group of students.

5. RÉSUMÉ

The considerations and the results of the students have (hopefully) shown: Openess and guidingness are not opposites in developing problem-based teaching resources for the fostering of mathematically gifted or interested primary school students. Instead, the aim must be to combine them appropriately, adapted to the level of experience of the respective group of students. This combination must always be critically evaluated, particularly with regard to whether the balance found for the current problem field can be changed in favor of openess for the next one.

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DIFFERENT INTERPRETATION OF ALGEBRAIC EXPRESSIONS

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Abstract

Students frequently make errors when solving algebraic problems and perceive algebra, often dubbed "the world of letters," as an isolated and impractical branch of elementary school mathematics. They feel that they are expected to manipulate symbols according to seemingly arbitrary rules, which they must learn without understanding. One source of substantial learning difficulties for many students when working with algebraic expressions may be connected to the concept of a procept. In mathematics education, the concept of procept refers to an integrated triad comprising a process, the mathematical object produced by that process, and a symbol that flexibly denotes either. The symbol does not constitute an additional entity; rather, it functions as a single notation that may represent a procedure in one context and an object in another. The authors report that this very fluidity — the shifting of a symbol between process and object — frequently contributes to students' misconceptions in algebra. Their findings indicate that many problem-solving difficulties arise from learners' struggles to coordinate this process-object duality and to interpret symbolic expressions appropriately within algebraic tasks.

Keywords: *procept, function, representations*

1. INTRODUCTION

Problem solving plays a central role in the development of mathematical thinking. Numerous studies have shown that engaging students in complex problem-solving tasks fosters deeper conceptual understanding and promotes the development of higher-order cognitive processes (Pólya, 1945; Schoenfeld, 1992; Lester, 2013). Effective support for this development involves designing learning environments that emphasize inquiry, reasoning, and reflection. Research consistently indicates that instructional strategies focused on metacognition, collaborative problem-solving, and open-ended tasks enhance students' mathematical reasoning and contribute to a more intuitive grasp of algebraic concepts (Boaler, 2016; Cai et al., 2019). Consequently, fostering problem-solving competence is essential for developing robust mathematical thinking that extends beyond routine computation toward genuine conceptual insight. Recent scholarship further highlights the importance of understanding how learners construct mathematical knowledge and how teachers can cultivate metacognitive and reasoning skills through carefully designed tasks and classroom practices. Influential contemporary contributions include Jo Boaler's research on mathematical mindsets and equitable learning environments, Jinfa Cai's work on problem-solving and reasoning, Alan Schoenfeld's analyses of mathematical thinking and instructional decision-making, the investigations of Anne Watson and John Mason into task design and learners' reasoning processes, and Paul Cobb's sociocultural examinations of classroom learning environments.

Collectively, these contributions continue to shape current perspectives on how mathematical thinking and problem-solving competence can be cultivated. Nonetheless, despite significant advances, research findings often fail to translate effectively into everyday classroom practice (Clarke & Hollingsworth, 2002; Cai & Hwang, 2023).

One persistent challenge in algebra learning is that students struggle to translate real-world scenarios into algebraic expressions. Many find it difficult to understand how symbolic forms correspond to quantities and relationships embedded in contextual problems, often resorting to procedural manipulation without grasping the meanings those symbols are intended to convey (Kieran, 1992; Kaput, 2008). This disconnect underscores the need for instructional approaches that explicitly link symbolic representations to underlying mathematical structures.

The present study illustrates how targeted instructional strategies can foster the development of students' algebraic thinking. The interventions integrated problem-oriented inquiry, multiple representations, and contextualized tasks designed to situate algebraic concepts within meaningful real-world scenarios. The methodology included analyzing students' performance on algebraic tasks, categorizing common error types (e.g., treating symbols merely as numbers or misapplying operations), examining students' explanations to identify points of conceptual breakdown, and designing as well as testing interventions — such as guided instruction, visual representations, interactive tools, and manipulatives — that explicitly support algebraic understanding. These strategies aimed to encourage students to interpret algebraic symbols as meaningful representations of mathematical relationships rather than as isolated procedural artifacts. While the notion of procept was developed by Tall and colleagues, Sfard's (1991) work on the process–object duality and reification address closely related aspects of how mathematical concepts evolve from actions to objects. The study also examined the impact of alternative instructional approaches, such as role-play and embodied learning with manipulatives, on students' conceptual development, and assessed whether contextualized activities promote deeper engagement with algebraic reasoning.

The expected outcomes included a more refined understanding of how and why students conceptualize algebra incorrectly, thereby informing the design of instructional materials that directly address common misconceptions. The study also sought to identify teaching strategies that effectively bridge the gap between procedural fluency and conceptual understanding, and to explore whether students' perceptions of algebra evolve as they begin to recognize its everyday relevance and its utility in interpreting abstract mathematical ideas.

2. THEORETICAL BACKGROUND

Tall (1998) expanded the notion that algebraic symbols — particularly letters — can simultaneously denote processes (operations) and mathematical objects (such as numbers or expressions). He argues that algebraic thinking requires students to navigate multiple representational states: the symbol as a process, the symbol as an object, and the symbol as a symbolic entity in its own right. One of his major contributions lies in elucidating the cognitive shift required for learners to transition from arithmetic, which is predominantly concrete and procedural, to algebra, which is inherently more abstract, structural, and symbolic. According to Tall, the development of algebraic thinking involves progressing through stages where symbols gradually acquire meanings that extend beyond their numeric or procedural origins. Tall (1998) also introduces the concept of cognitive conflict, which arises when students encounter contradictions in their understanding while attempting to grasp new mathematical ideas. In algebra, such conflict occurs when learners attempt to interpret symbols simultaneously as processes and objects, potentialyly resulting in confusion or frustration if the relationship between these roles is not fully understood. These insights help explain why students may struggle with problem-solving and fail to comprehend algebraic rules and structures. Moreover, Tall introduced the concept of symbolic forms, describing how mathematical symbols are used to represent both objects and processes in a context-dependent manner. He advocates for the use of visual and dynamic representations to bridge the gap between concrete and abstract thinking, providing a foundation for instructional strategies that explicitly support the cognitive transitions required to understand algebraic procepts. Interventions and teaching methods informed by these principles aim to guide students toward a more integrated comprehension of algebraic symbols. Booth (1984) focused on the transition from concrete to abstract thinking in mathematics and examined the difficulties students encounter when engaging with abstract algebraic concepts. His research on the role of variables in algebra provides insight into why students often perceive algebra as arbitrary and disconnected from real-world meaning, aligning with observations of students treating algebra as "the world of letters." Sfard (1991) emphasized the dual nature of mathematical concepts, highlighting that mathematical symbols, including algebraic symbols, function both as static objects and dynamic processes. Her work on commognitive (communication-based) frameworks aligns with the concept of procepts, which inherently embody both process and object. Sfard further introduced the notions of "reification" (the transformation of a process into an object) and "conceptual blending," both of which offer a theoretical basis for understanding algebraic symbols as simultaneously processes and objects.

Fuson (1992) investigated the role of conceptual understanding in mathematics learning, particularly examining how mastery of arithmetic processes and operations influences algebraic learning. She emphasized the importance of process-object fusion, especially in relation to operations and algebraic rules. Kaput (1999) contributed significantly to the understanding of algebra teaching and learning, focusing on the development of algebraic thinking and the challenges students face in mastering it, including misconceptions regarding symbolic manipulation. Radford (2006) highlighted the cognitive and epistemological significance of procepts in algebra and mathematics education, demonstrating how students' understanding of algebraic symbols can be simultaneously process and object-based, thereby challenging traditional purely symbolic approaches to teaching algebra. Harel and Sowder (2007) reviewed how students comprehend key algebraic concepts, such as variables, operations, and expressions, which are central to the notion of procepts. Their work emphasizes the cognitive development of algebraic thinking and identifies common student misconceptions, crucial for addressing issues in symbolic manipulation. More recently, Sun, Sun, and Xu (2023) conducted a cross-sectional study with elementary students (grades 3–5) to investigate the early development of algebraic thinking. They assessed three dimensions: generalized arithmetic, functional thinking, and quantitative reasoning. Using latent class analysis and interviews, they identified three developmental trajectories: from arithmetic thinking to concrete algebraic thinking, and ultimately to symbolic algebraic thinking. As students progressed, their ability to generalize and use symbolic representations increased. The study underscores the importance of deliberately integrating generalization skills and multiple representations, including symbols, into elementary curricula to support early algebra instruction.

3. RESEARCH PROBLEM

The statement that initially motivated this research is that in teaching algebra, concrete (enactive) and visual (iconic) representations — as originally conceptualized by Bruner (1966) — should be sustained longer and used more deliberately to support students' transition from arithmetic to algebraic thinking (Stankov, 2005). During teaching practice, we repeatedly come across the challenges students face while translating word problems into algebraic language. In 7th grade the following word problem is commonly presented: "Steven had a certain number of marbles and John had double that amount plus 3. They had 15 altogether. How many marbles did they have each?". The task is to write down the equation according to the information, solve it and verify the result. The most common errors that can be recognized are of different kinds. Some students solve the problem in their heads, without writing down the process, thus they applied mental arithmetic, in their approach writing down anything is nonsense. Some of them consistently use rules they come up with that are logically incorrect. And some use random rules inconsistently. Naturally the two latter groups cannot achieve the right result. To correctly formulate and solve the previously described problem, it is essential to have a clear understanding of the concepts, rules, relations, and notational conventions that enable the construction of the corresponding equation, as solving the task ultimately entails formulating equations that comprise variables, algebraic operations, and relational structures. A central aim of the intervention was not only to introduce students to the structural and procedural dimensions of mathematics, but also to foreground its capacity to model, organize, and deepen the understanding of everyday phenomena, thereby supporting the abstraction of algebraic concepts. To achieve this, algebraic ideas were situated in relation to both tangible and representational elements — including manipulatives, symbolic notations, and practical tools — while being embedded within learning experiences that encouraged creative exploration, expressive reasoning, and reflective engagement. This dual emphasis sought to render abstract mathematical ideas more accessible by linking them to students' prior knowledge, perceptual experiences, and evolving cognitive frameworks. The initial stage of the study involved the design of learning tasks and the establishment of a pedagogical environment capable of supporting students' engagement with algebraic concepts from multiple perspectives, with particular attention to the notion of procept, understood as the dual nature of mathematical symbols functioning simultaneously as processes and concepts (Tall & Vinner, 1981). To ensure alignment with national curriculum standards and pedagogical best practices, the intervention was designed with reference to the framework outlined in the Hungarian National Curriculum, which emphasizes the gradual development of mathematical thinking — including the shift from arithmetic to algebraic and abstract reasoning — through spirally structured content, age-appropriate cognitive growth, and a blend of reproductive and creative problem-solving skills (EMMI, 2020; NEFMI, 2012).

Moreover, implementing the intervention within a Waldorf school context further supports its goals: the pedagogical philosophy of these schools advocates for a holistic, multimodal approach to mathematics learning, encouraging the use of manipulatives, visual representations, and a progression from concrete experience toward abstraction (Hungarian Waldorf Schools Federation, 2013).

In this study, eight 6th-grade students of average academic ability participated in the intervention. The program comprised four instructional sessions, each lasting approximately two hours, for a total of eight hours of instructional time. Activities were carefully designed to facilitate transitions among enactive, iconic, and symbolic modes of representation, in accordance with Bruner's (1966) representational framework. Tasks integrated everyday symbols, object-based manipulations, guided discussions, visual modeling, role-play, and structured opportunities for students to articulate the relationships inherent in mathematical situations. Learners worked in small groups or pairs, promoting collaboration, shared reasoning, and the co-construction of understanding. This arrangement encouraged students to externalize their thinking, negotiate meaning, pose conjectures, and refine symbolic expressions grounded in contextualized experiences, while simultaneously promoting their problem-solving skills in alignment with Pólya's framework. The qualitative design — combining guided discussions, hands-on activities, role-play, and analysis of symbols drawn from everyday life — enabled systematic observation of students' mathematical reasoning, conceptual development, and strategic problem-solving within a context less constrained by conventional instruction. Context-rich, multimodal tasks proved particularly effective in fostering the internalization of variable-based reasoning and symbolic thinking, while also providing opportunities for learners to apply, adapt, and reflect on solution strategies. The collaborative, group- or pair-based setting, supported by observational records, detailed field notes, and post-activity assessments, allowed for detailed tracking of how students identified, manipulated, and transferred algebraic reasoning across diverse contexts, integrating both conceptual understanding and problem-solving competence.

3.1. Algebraic notions and the procept

Next sections present a series of intervention-based activities designed to support students in developing a flexible understanding of variables and mathematical symbols, with particular attention to the notion of procept.

3.1.1. Concept of variables

The first stage of the intervention aimed to broaden students' understanding of variables beyond traditional classroom instruction. Students engaged in guided discussions about changes in forms and colors in nature, such as minerals, plants, and animals, focusing on temporal and spatial transformations, as well as on human-created and human-directed transformational processes through which functional objects are produced. These discussions established a foundation for creative tasks in which students constructed narratives emphasizing the transformations of shapes and forms. Students worked with manipulatives, including clay, playdough, and flexible, shape-retaining cords, and drew inspiration from animated media to visualize dynamic changes (for instance Barbapapa). Building on these experiences, subsequent activities were explicitly designed to support students' engagement with mathematical symbols and algebraic concepts, with particular attention to the notion of procept, that is, the dual nature of symbols as both processes and objects. By connecting the perceptual and manipulative experiences from earlier stages to symbolic representations, students were encouraged to recognize and interpret mathematical transformations in a manner analogous to the dynamic changes observed in the natural and human-designed world. This approach provided a coherent bridge from concrete, context-rich exploration to the abstraction required for understanding variables, operations, and the structural relationships central to algebraic reasoning (Figure 1).

Figure 1. *Different forms*

To deepen engagement with the dynamic nature of variables, students' narratives were incorporated into role-play activities. Building on these narratives, students were encouraged to reflect on how each character's actions could be represented symbolically, highlighting the dual role of symbols as both processes and objects in algebraic thinking. Individual students enacted multiple characters, while multiple students portrayed the same character, thus embodying the interchangeability and variability of algebraic concepts through performance. This embodied approach allowed learners to experience firsthand the concept of procept, linking concrete enactment to abstract representation (Figure 2).

Figure 2. *Role-plays*

A subsequent activity required students to construct spatial conundrums, replacing personal identifiers with positional references. Specifically, students described peers' locations in the classroom relative to fixed reference points, such as the door or specific chairs, rather than using names. This exercise emphasized relational reasoning and reinforced the concept of variables as dynamic entities. Through these spatial tasks, students began to translate observed relationships into symbolic notation, connecting concrete positional reasoning to algebraic symbols and enhancing their understanding of variables as flexible, relational entities.

3.1.2. From Role-Play to Acting Out Function Machines

In mathematics, a function machine model is a conceptual tool that is applied to understand functions. Building on the earlier role-play experiences, this model provided a tangible means for students to connect symbolic representations with observable transformations, thereby reinforcing the dual nature of algebraic symbols as both processes and objects (procepts). It offers a simple way to visualize how functions transform inputs into outputs. By situating this visualization within familiar contexts, learners could relate the abstract concept of a function to concrete, real-world transformations, enhancing both engagement and conceptual clarity. As functions involve variables, the intervention leveraged students' prior role-play experience to enact the operation of function machines with different variables. Drawing on their enactments of dynamic characters and positional reasoning, students embodied the variability inherent in functions, experiencing firsthand how changing inputs systematically produced corresponding outputs. Initially, different kinds of everyday household devices — both mechanical and electronic (e.g., grater, grinder, juicer) that process food (e.g., nuts, fruits) — were introduced to students. After examining their operating principles, learners formed small groups to explore how the input was transformed into output. They then recorded the input and output elements in tables, creating visual representations of variables. This activity scaffolded the transition from concrete manipulation to symbolic notation, allowing students to visually map relationships and internalize the functional behavior of variables. Continuing with the role-play, the next challenge involved operating a function machine model constructed from cardboard and plastic tubing. Through this hands-on enactment, students could physically experience quantitative changes, bridging embodied understanding with abstract algebraic reasoning. This multimodal approach promoted a deeper appreciation of variables as flexible, context-dependent entities while simultaneously reinforcing the concept of procept, demonstrating that the same symbols can represent both the operational process and the resultant object, consistent with the function machine framework (Hourigan, Leavy, & McMahon, 2012).

3.1.3. The procept with respect to symbols from the everyday environment

Mediated discussions guided students in analyzing printed symbols encountered in everyday life, examining how these symbols could represent input, output, process, or combinations thereof. Building on their prior experiences with role-play and function machines, these discussions helped students bridge concrete, embodied understanding with symbolic representation, reinforcing the dual nature of symbols as both processes and objects. Students categorized symbols into five types: those representing only the input, only the outcome, both process and outcome, input-process-outcome, or none explicitly. This classification activity encouraged learners to systematically interpret symbolic information, linking perceptual cues from familiar contexts to abstract structures. These activities aimed to strengthen students' ability to recognize abstract ideas embedded in familiar, concrete contexts. By situating symbols within meaningful, real-world scenarios and connecting them to prior enactments and manipulations, students could more readily perceive variables as dynamic and relational entities. This approach supported the internalization of complex mathematical structures, including functions and variable-based reasoning, in alignment with the concept of procept. By connecting symbols to everyday experiences, students could integrate their perceptual understanding with algebraic reasoning, supporting the internalization of complex mathematical structures. Observational data documented students' engagement, interpretive strategies, and ability to relate symbols to underlying processes, providing evidence for the effectiveness of multimodal, context-rich interventions in supporting conceptual development. These findings indicate that integrating embodied, visual, and symbolic modalities not only enhances procedural fluency but also fosters a deeper, conceptually grounded understanding of algebraic relationships, laying the foundation for flexible and transferable mathematical thinking.

3.1.4. The procept with respect to mathematical symbols

As students progressed toward formal use of mathematical symbols, they recorded the processes and outcomes experienced during function-machine activities in structured tables. By translating these embodied and visual experiences into tabular representations, learners were able to clearly differentiate between operational processes and their corresponding results, while simultaneously connecting these relationships to symbolic notation. This method provided a tangible bridge between concrete, experiential understanding and abstract representation, thereby promoting awareness of the dual nature of algebraic symbols as both processes and object*s* (procepts). On the path toward using mathematics symbols, students used table forms to write down the process and the result experienced in working with function machines (Table1.).

Table 1. *Table for function machine*

	symbol for result	process
arithmetics	3	2+1
arithmetics	6	5+1
arithmetics	113	112+1
algebra	sz+1	sz+1

In this way, they highlighted the distinction between a process and its result, as well as the corresponding symbolic notation. Building on this foundation, subsequent tasks were carefully designed to extend students' capacity for flexible reasoning, enabling them to shift seamlessly between numerical operations, symbolic expressions, and their associated meanings. For example, students were asked to: "Write in two ways: multiply 2 and 4, and then add the number to this product"; "Write the number that is 2 greater than three times four"; or "If John has n apples and Anne has two more, and together they have 14 apples, how many does each have?" (Table 2).

Table 2. *Table both with concrete numbers and variable*

John	process	Anne
1	+2	3
2	+2	4
3	+2	5
n	+2	n+2

In engaging with these tasks, students applied variables and interpreted symbolic expressions with increasing fluency, drawing on skills and conceptual understanding developed through the prior role-play, manipulative, and function-machine activities. By connecting word-problem contexts to formal symbolic representations, learners not only strengthened procedural competence but also deepened their conceptual grasp of algebra as a coherent and meaningful system. Furthermore, these activities reinforced the notion that symbols simultaneously encapsulate processes and objects, supporting the internalization of abstract mathematical relationships and the development of flexible, transferable reasoning strategies essential for early algebraic thinking.

3.2. Findings

The primary aim of this study was to explore how multimodal, context-rich mathematical activities can support the development of pre-algebraic reasoning in sixth-grade students. The study addressed three overarching research questions: How do students transition from concrete, enactive representations to iconic and symbolic representations in early algebraic reasoning? In what ways do multimodal activities foster relational and structural understanding of algebraic concepts, including the dual nature of symbols as both processes and objects (procepts)? How does engagement in context-rich, collaborative problem-solving tasks support the development of variable-based generalization and flexible symbolic reasoning?

Analysis of observational data, field notes, and post-activity assessments revealed consistent patterns in students' early algebraic reasoning. Learners gradually progressed from manipulating tangible objects and enacting transformations through role-play to interpreting symbolic relationships and constructing equations. Tabular representations and function-machine exercises acted as bridges between experiential understanding and abstract notation, supporting recognition of the dual nature of symbols (procepts). Students increasingly demonstrated that equations represent relationships rather than simple computations, articulated the equivalence of different expressions, and recognized equality as a relational concept. Misconceptions related to the equal sign — such as interpreting it solely as "the answer comes after the =" — diminished through balancing tasks and role-plays embodying equilibrium. Furthermore, students showed growing capacity to generalize patterns and transfer reasoning across contexts, employing variables to represent unknown quantities and flexibly interpreting symbolic expressions. These findings indicate that even prior to formal algebra instruction, students can engage with symbolic, relational, and generalized reasoning when supported by multimodal, context-rich learning experiences, providing a conceptual foundation for later algebraic learning rather than relying solely on procedural knowledge.

4. CONCLUSION

The study demonstrates that multimodal, context-rich mathematical activities can effectively support the development of algebraic reasoning in sixth-grade students. Through a carefully structured intervention, learners engaged with concrete, iconic, and symbolic representations, explored functional relationships, and enacted variable-based reasoning in collaborative, experiential settings. Observational data, field notes, and post-activity assessments revealed that students internalized the dual nature of algebraic symbols (procepts), successfully transitioned between representations, recognized relational structures, and generalized patterns across contexts — even prior to formal algebra instruction.

These findings highlight the importance of integrating multimodal experiences, role-play, and authentic contexts into mathematics teaching. Such approaches provide learners with opportunities to construct meaning, articulate their reasoning, and collaboratively negotiate mathematical relationships. By situating algebraic concepts within tangible and representational frameworks, educators can support the gradual abstraction of mathematical ideas while maintaining strong connections to students' prior knowledge and lived experiences. Furthermore, designing tasks in line with Pólya's problem-solving principles encourages students to analyze problems, develop strategies, carry out solutions, and reflect on outcomes, thereby reinforcing both procedural skills and deep conceptual understanding.

Looking ahead, the study suggests multiple avenues for future research and educational practice. Longitudinal studies could investigate the lasting impact of early algebra interventions on subsequent achievement and conceptual development. Additional research might explore how technology-mediated tools, digital manipulatives, or cross-curricular applications can enhance multimodal learning of algebraic concepts. Extending this approach to diverse educational contexts could also shed light on how sociocultural and pedagogical factors influence the development of algebraic thinking and problem-solving skills. Ultimately, a sustained focus on context-rich, student-centered, and multimodal instructional strategies — rooted in both algebraic concepts and established problem-solving frameworks — holds considerable potential to transform algebra education, fostering deeper understanding, flexible reasoning, and meaningful engagement with mathematics.

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